revision

If $f(x) = \begin{cases} x - 1, & \text{as } x \geq 2 \\ k - x, & \text{as } x < 2 \end{cases}$

is continuous at x = 2, find the value of k then discuss the differentiability of f(x) at x = 2.

Solution

$f(2^+) = 2 - 1 = 1$

$f(2^-) = k - 2$

Since f(x) is continuous at x = 2

therefore $f(2^+) = f(2^-)$

therefore $k - 2 = 1$ or $k = 3$

therefore $f(x) = \begin{cases} x - 1, & \text{as } x \geq 2 \\ 3 - x, & \text{as } x < 2 \end{cases}$

$f'(2^+) = \lim_{h \to 0^+} \frac{(2+h-1) - 1}{h} = 1$

$f'(2^-) = \lim_{h \to 0^-} \frac{(3-2-h) - 1}{h} = -1$

If y = 2sin x - xcos x , prove that $\frac{d^2y}{dx^2} + y = 2\sin x$

Solution

$\because y = 2\sin x - x\cos x \therefore \frac{dy}{dx} = 2\cos x + x\sin x - \cos x$

& so $\frac{dy}{dx}$ $\cos x + x\sin x$

$\because \frac{d^2y}{dx^2} = -\sin x + x\cos x + \sin x$ *i.e.* $\frac{d^2y}{dx^2} = x\cos x$

From the header of the problem $x\cos x = 2\sin x - y$

$\therefore \frac{d^2y}{dx^2} = 2\sin x - y$ $\frac{d^2y}{dx^2} + y = 2\sin x.$

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If $xy = 1 + x^2$, prove that $x \frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2$

Solution

$\because xy = 1 + x^2 \quad \therefore x\frac{dy}{dx} + y = 2x$

& so $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = 2$

$\therefore x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2.$

Find the equation of the normal to the curve $x^2 - xy = 6$ at the point (3 , 1)

Solution

$2x - (x\frac{dy}{dx} + y) = 0$

$2(3) - (3m + 1) = 0$

$6 - 3m - 1 = 0$

$\therefore 3m = 5 \qquad \therefore m = \frac{5}{3}$

The equation of the normal is:

$y - 1 = \frac{-3}{5}(x - 3)$

$y = \frac{-3}{5}x + \frac{14}{5}$

A particle moves along the curve $x^2 + y^2 + 2x - y - 10 = 0$.

Find the position of the particle at the instant when $\frac{dy}{dt} = 2\frac{dx}{dt}$

Solution

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2\frac{dx}{dt} - \frac{dy}{dt} = 0$

$\therefore 2x\frac{dx}{dt} + 4y\frac{dx}{dt} = 0 \Rightarrow (2x + 4y)\frac{dx}{dt} = 0$

$\therefore 2x + 4y = 0 \Rightarrow x = -2y$

$4y^2 + y^2 - 4y - y - 10 = 0$

$5y^2 - 5y - 10 = 0 \Rightarrow y^2 - y - 2 = 0$

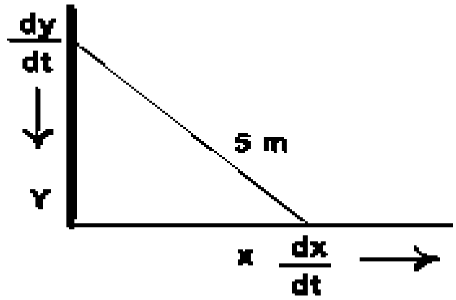
$\therefore y = 2$ *or* -1 *& so* $= -4$ *or* 2

the points are (2 , -1) & (-4 , 2)

A ladder of length 5m rests with one of its ends on a horizontal floor and with the other end against a vertical wall.

If the lower end slides away from the wall at rate 1m/sec, find the rate of descent of the upper end when the lower end becomes 3m distant from the wall.

Solution



Since $y^2 + x^2 = 25$,

$\therefore\ 2y\dfrac{dy}{dt} + 2x\dfrac{dx}{dt} = 0 \qquad y\dfrac{dy}{dt} + x\dfrac{dx}{dt} = 0$

$y\dfrac{dy}{dt} = -x\dfrac{dx}{dt}$

Since $x = 3$

$\therefore\ y\dfrac{dy}{dt} = -3\dfrac{dx}{dt}$ *(I)* and $y^2 + (3)^2 = 25$ $\quad y^2 = 25 - 9 = 16$ or $y = 4$

Since the lower end slides away from the wall at rate 1m/sec

$\therefore\ \dfrac{dx}{dt} = 1$ *(III)*

From *(I)*, *(II)* and *(III)* we get $4\dfrac{dy}{dt} = -3(1)$ or $\dfrac{dy}{dt} = -\dfrac{3}{4}$ m/sec

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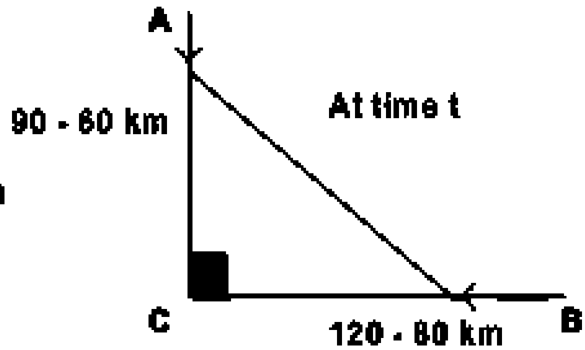
AC and BC are two orthogonal roads where AC = 90km and BC = 120km.

A car moved from A towards C with velocity 60km/h and at the same moment another car moved from B towards C with velocity 80km/h
Find the rate of change of the distance between the two cars just after one hour.

Solution



$S^2 = (90 - 60t)^2 + (120 - 80t)^2$

As $t = 1$ hour, $S^2 = (30)^2 + (40)^2 = 2500$ or $S = 50$Km

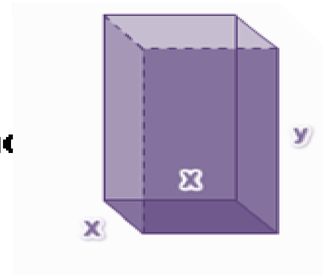
$2S\dfrac{ds}{dt} = 2(90 - 60t)(-60) + 2(120 - 80t)(-80)$

$2(50)\left(\dfrac{ds}{dt}\right)_{t=1} = 2(30)(-60) + 2(40)(-80)$

$100\left(\dfrac{ds}{dt}\right)_{t=1} = -3600 - 6400 = -10000$

$\therefore\ \left(\dfrac{ds}{dt}\right)_{t=1} = -100$km/h

Find the greatest volume of a cuboid with a square base and total surface area 600 cm^2.



Solution

Let the dimensions of the cuboid with a square base be x, x and y

Total surface area = $2(x^2 + xy + xy) = 600$ or $x^2 + 2xy = 300$

$\therefore y = \dfrac{300 - x^2}{2x}$ (I)

$\therefore$ Volume $V = (x)(x)\left(\dfrac{300 - x^2}{2x}\right) = 150x - \dfrac{x^3}{2}$

$\dfrac{dv}{dx} = 150 - \dfrac{3x^2}{2}$

$\dfrac{dv}{dx} = 0$ implies $\dfrac{x^3}{2} = 150$ or $x^2 = 100$ or $x = 10$

Substituting in (I) we get y = 10

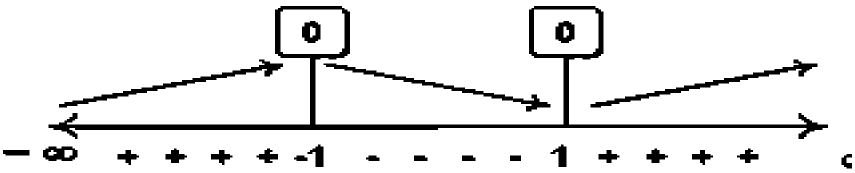
i.e. the maximum volume = 1000 cm^3

Sketch the graph of the function: $f(x) = x^3 - 3x + 2$

Solution

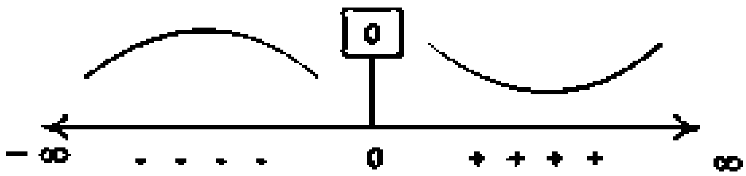
Step 1:

$f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$
$f'(x) = 0$ when $x = -1$ or 1
when $x = -1$, $f(x) = 4$,
and when $x = 1$, $f(x) = 0$
therefore:
- at the point (- 1 , 4) there is a local maximum value.
- at the point (1 , 0) there is a local minimum value.



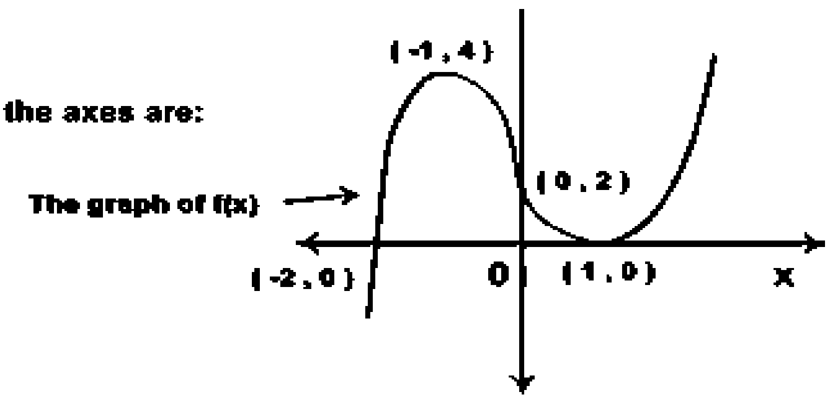
Step 2:

$f''(x) = 6x$
$f''(x) = 0$ when $x = 0$
when $x = 0$, $f(x) = 2$
therefore the point (0 , 2) is an inflection point.



Step 3:

The points of intersections with the axes are:
(-2 , 0), (0 , 2) and (1 , 0)



$$\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$, provided that $n \neq -1$, and a , b are constants .

1. $\int (2x+1)^6\, dx = \frac{(2x+1)^7}{2(7)} + C = \frac{(2x+1)^7}{14} + C$

2. $\int (5-4x)^9\, dx = \frac{(5-4x)^{10}}{-4(10)} + C = \frac{-(5-4x)^{10}}{40} + C$

3. $\int \sqrt{(1+6x)}\, dx = \int (1+6x)^{1/2}\, dx = \frac{(1+6x)^{3/2}}{6(3/2)} + C$

$$= \frac{m\ (1+6x)^{3/2}}{9} + C$$

4. $\int x^7 \left(1-\frac{1}{x}\right)^7 dx = \int \left(x\left(1-\frac{1}{x}\right)\right)^7 dx = \int (x-1)^7\, dx$

$$= \frac{(x-1)^8}{8} + C$$

1. $\int \sin x\, dx = -\cos x + C$
2. $\int \cos x\, dx = \sin x + C$
3. $\int \sec^2 x\, dx = \tan x + C$

Where C is an arbitrary constant, The proof is direct by differentiating the right hand side.

4. $\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + C$
5. $\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + C$
6. $\int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + C$

Where C is an arbitrary constant, The proof is direct by differentiating the right hand side.

Examples :

1. $\int (\cos x - \sin x)\, dx = \sin x + \cos x + C$
2. $\int (\sec^2 x + \cos x)\, dx = \tan x + \sin x + C$
3. $\int \cos(2x+3)\, dx = \frac{1}{2}\sin(2x+3) + C$
4. $\int \sec^2\left(\frac{x}{2}+1\right) dx = 2\tan\left(\frac{x}{2}+1\right) + C$

Evaluate $\int (1+\sin x)^2 \ dx$

Solution :

$(1+\sin x)^2 = 1 + 2 \sin x + \sin^2 x$

as $\cos 2x = 1 - 2\sin^2 x$

then $2\sin^2 x = 1 - \cos 2x$ i.e $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$

$\therefore \int (1+\sin x)^2 \ dx = \int (1 + 2\sin x + \frac{1}{2} - \frac{1}{2}\cos 2x)\ dx$

$= \int (\frac{3}{2} - \frac{1}{2}\cos 2x + 2\sin x)\ dx$

$= \frac{3}{2} x - \frac{1}{4} \sin 2x - 2\cos x + C$

Find the equation of the curve $y = f(x)$, given that $y'' = 2\cos 2x$ and the equation of the tangent to the curve at the point $(0,1)$ is $y = x + 2$.

$\int y'' \ dx = y'$

(as the derivative of the R.H.S. with respect to x is y'') .

$\therefore y' = \int 2\cos 2x \ dx = \sin 2x + C$ (C constant)

as $(y')_{x=0} = 1$

Thus $0 + C = 1$ i.e $C = 1$

$y' = \sin 2x + 1$

$y = \int (\sin 2x + 1)\ dx = -\frac{1}{2}\cos 2x + x + A$ (A constant)

As the curve passes through the point (0,1) weget

$1 = -\frac{1}{2} + 0 + A$ i.e $A = \frac{3}{2}$

Hence the equation of the required curve is

$y = x - \frac{1}{2} \cos 2x + \frac{3}{2}$

Find the equation of the curve $y = f(x)$ given that, $f''(x) = 6x$ and the tangent to this curve at the point (1 , 4) is the straight line $y = -2x + 6$.

Solution :

As $f''(x) = 6x$, then $f'(x) = 3x^2 + C_1$. Now $f'(x)$ represents the slope of the tangent to the curve at any point on it with abscissa x . As the line $y = -2x + 6$ is tangent to the curve at $x = 1$, then $f'(1)$ equals the slope of the line, i.e. $f'(1) = -2$. From this equality we evaluate C_1:

$$-2 = 3(1^2) + C_1 \quad \Rightarrow \quad C_1 = -5$$

Thus , $f'(x) = 3x^2 - 5$.

Integrating once again, we get

$$f(x) = \int (3x^2 - 5)\, dx = x^3 - 5x + C_2$$

But as the curve passes through the point (1, 4) (that is $y = 4$ when $x = 1$) we get

$$4 = 1 - 5 + C_2 \Rightarrow C_2 = 8$$

Thus the equation of the curve is

$$y = x^3 - 5x + 8 .$$

Find the equation of the curve $y = f(x)$ given that $(dy/dx) = [(2x - 3)/(5 - 2y)]$ and the curve passes through the point (1, 2) .

$$\int (5 - 2y) \frac{dy}{dx}\, dx = \int (2x - 3)\, dx$$

$$5y - y^2 = x^2 - 3x + C \quad (C \text{ constant})$$

As the curve passes through the point (1 , 2) , it satisfies its equation.

Hence ,

$$5(2) - 2^2 = 1^2 - 3(1) + C \Rightarrow C = 8$$

Thus, the equation of the required curve is

$$5y - y^2 = x^2 - 3x + 8 .$$

If the slope of the tangent to a curve at any point (x, y) on it is given by $(dy/dx) = 3x^2 - 6x - 9$ and if the curve has a local maximum value equals 10, find the equation of the curve and the local minimum value if it exists .

Let $y = f(x)$ be the equation of the curve, then $(dy/dx) = 3x^2 - 6x - 9 = 3(x+1)(x-3)$ (1)

Now (dy/dx) equals zero at $x = -1$, $x = 3$. Differentiating again, we get $(d^2y/dx^2) = 6x - 6 = 6(x-1)$.

Evaluating the second derivative at $x = -1$, and at $x = 3$, we get

$$(d^2y/dx^2)\Big|_{x=-1} = 6(x-1)\Big|_{x=-1} = -12 < 0$$

Thus there is a local maximum value equals 10 at $x = -1$.

Hence the curve passes through the point $(-1, 10)$.

To find the equation of the curve, we integrate both sides of equation (1) w.r. to x, to get

$$y = x^3 - 3x^2 - 9x + C$$

We evaluate the constant C by putting $x = -1$, $y = 10$ Thus, $C = 5$, and the equation of the curve is

$$y = x^3 - 3x^2 - 9x + 5 .$$

This curve has a local minimum value at $x = 3$ and equals

$$y\Big|_{x=3} = 27 - 27 - 27 + 5 = -22 .$$