School Book's Exams
&
Model Answers
Calculus
Sec.3

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School book exams

3rd Sec .Calculus

Test 1

1] Choose the correct answer

1- Which of the following functions satisfies the relation $\frac{d^3y}{dx^3} = y$

a] $y = \frac{1}{12}(x+1)^4$ b] $y = \sin x$

c] $y = e^x$ d] $y = \frac{x}{x-4}$

2- If the radius length of a circle increases at a rate $\frac{1}{\pi}$ cm/sec , the circumference of the Circle increases at a rate of .... cm/sec

a] $\frac{2}{\pi}$ b] 2 c] π d] 2π

3- The curve of the function f where $f(x) = x^3 - 3x^2 + 2$ is convex upwards when $x \in$

a) $]-\infty, 0[$ b) $]-\infty, 1[$ c) $]1, 3[$ d) $]1, \infty[$

4- $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (\sin x + \cos x)\, dx$ equals

a] 4 b] 2 c] Zero d] π

5- If f is a continuous function on R . $\int_3^5 2f(x)dx = 8$, $\int_3^4 3f(x)dx = 9$ then $\int_4^5 5f(x)dx =$ ………………………

a] zero　　　b] 1　　　c] 3　　　d] 5

6- The area of the region bounded by the curve $y = \sqrt{16 - x^2}$ and x –axis approximated is square units equals :

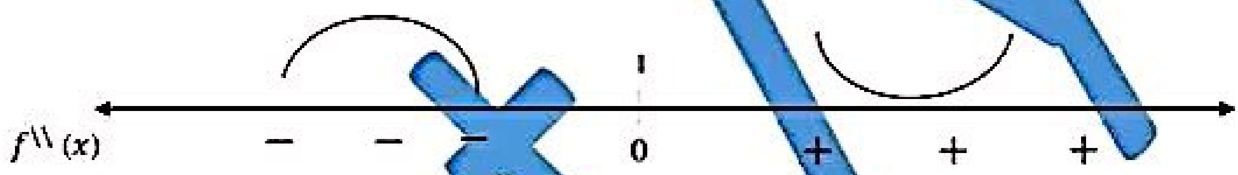
a] 16π　　　b] 12π　　　c] 8π　　　d] 4π

The Solution

1- c) $y = e^x$　　$\therefore \frac{dy}{dx} = e^x$, $\frac{d^2y}{dx^2} = e^x$, $\frac{d^3y}{dx^3} = e^x$

2- $c = 2\pi r$　　$\therefore \frac{dC}{dt} = 2\pi \frac{dr}{dt} = 2\pi \times \frac{1}{\pi} = 2$

3- $f`(x) = 3x^2 - 6x$　　$\therefore f``(x) = 6x - 6$　　Let $f``(x) = 0$ $\therefore x = 1$



$\therefore$ The curve convex up at $]-\infty, 1[$

4- $[-\cos x + \sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = [-\cos 90 + \sin 90] - [-\cos(-90) + \sin(-90)] = (0 + 1) - (0 - 1) = 2$

5- $\because \int_3^5 2f(x)dx = 8$ $\therefore \int_3^5 f(x)dx = 4$

$\because \int_3^4 3f(x)dx = 9$ $\therefore \int_3^4 f(x)dx = 3$

$\therefore \int_4^5 f(x)dx = \int_3^5 f(x)dx - \int_3^4 f(x)dx = 4 - 3 = 1$　　$\therefore \int_4^5 5f(x)dx = 5$

6 - The direct solution $\because y = \sqrt{16 - x^2}$ by squaring both sides

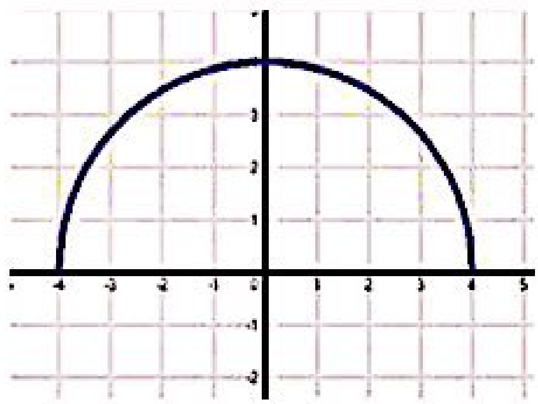
$\therefore y^2 = 16 - x^2$　　$\therefore x^2 + y^2 = 16$

and this is the equation of a circle its center is the origin point and its radius = 4 unit length

$\therefore$ Its area = 16π unit area

$\therefore y = \sqrt{16 - x^2}$ represent a semi circle

$\therefore$ Its area = 8π unit area



Another solution :

Let $y = 0 \therefore 16 - x^2 = 0$ $\therefore x = 4$ or $x = -4$

$\therefore$ The required area $A = \int_{-4}^{4} \sqrt{16 - x^2}\ dx = 2\int_{0}^{4} \sqrt{16 - x^2}\ dx$

Let $x = 4 \operatorname{Sin} y$ $\therefore dx = 4 \operatorname{Cos} y\ dy$ at $x = 0 , \therefore y = 0$ at $x = 4 \therefore y = \frac{\pi}{2}$

$\therefore 16 - x^2 = 16 - 16Sin^2 y = 16(1 - Sin^2\theta) = 16\, Cos^2\theta$ $\therefore \sqrt{16 - x^2} = 4 \operatorname{Cos}\theta$

$\therefore A = 2\int_{0}^{\frac{\pi}{2}} 4 \operatorname{Cos}\theta \times 4 \operatorname{Cos}\theta\ d\theta = 16\int_{0}^{\frac{\pi}{2}} 2\, Cos^2\theta\ d\theta = 16\int_{0}^{\frac{\pi}{2}} (\operatorname{Cos} 2\theta + 1)\ d\theta$

$= 16\left[\frac{1}{2}\ Sin 2\theta + \theta\right]_{0}^{\frac{\pi}{2}} = 16\left(\left[\frac{1}{2}\operatorname{Sin} 180 + \frac{\pi}{2}\right] - \left[\frac{1}{2}\operatorname{Sin} 0 + 0\right]\right) = 8\pi$ unit area .

2-a] Find : $\int Sin\, x\ \operatorname{Cos}^3 x\ dx$

The Solution

$I = \frac{-1}{4} \operatorname{Cos}^4 x + C$

2-b] If $e^{xy} - x^2 + y^3 = 0$ **then find** $\frac{dy}{dx}$ when $x = 0$

The Solution

at $x = 0$ $\therefore e^0 - 0 + y^3 = 0$, $\therefore y^3 = -1$, $\therefore y = -1$

$e^{xy}\left(x\frac{dy}{dx} + y\right) - 2x + 3y^2\frac{dy}{dx} = 0$ at $x = 0$, $y = -1$ $\therefore e^0[0 - 1] - 2(0) + 3(-1)^2\frac{dy}{dx} = 0$

$-1 + 3\frac{dy}{dx} = 0$ $\therefore 3\frac{dy}{dx} = 1$ $\therefore \frac{dy}{dx} = \frac{1}{3}$

3-a] Find the equation of the tangent to the $x^2 - 3xy - y^2 + 3 = 0$ at point $(-1 , 4)$.

The Solution

$\because x^2 - 3xy - y^2 + 3 = 0$ $\therefore$ $2x - 3y - 3x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$ at $(-1 , 4)$

$\therefore 2(-1) - 3(4) - 3(-1)\frac{dy}{dx} - 2(4)\frac{dy}{dx} = 0$ $\therefore -2 - 12 + 3\frac{dy}{dx} - 8\frac{dy}{dx} = 0 \therefore -14 - 5\frac{dy}{dx} = 0$

$\therefore -14 = 5\frac{dy}{dx}$ $\therefore$ $\frac{dy}{dx} = \frac{-14}{5}$ $\because$ **The equation of the tangent** $\frac{y - y_1}{x - x_1} = m$ $\therefore \frac{y - 4}{x - (-1)} = \frac{-14}{5}$

$\therefore 5(y - 4) = -14(x + 1)$ $\therefore 5y - 20 = -14x - 14$ $\therefore 14x + 5y - 6 = 0$

3-b] The lengths of the legs of the right angle a right – angled triangle at a moment , are 6 cm and 30 cm. If the length of the first leg increases at a rate of $\frac{1}{3}$ cm/min and the length of the second leg decreases at a rate of 1 cm/min , **Find:**

1– The rate of increase in the area of the triangle after 3 minutes

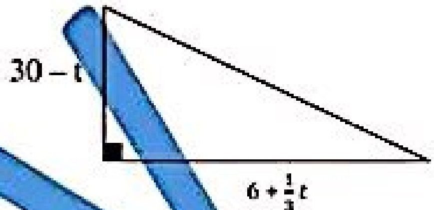
2– The time at which the increase of the area of the triangle stops.

The Solution

The length of the two legs at any time $6+\frac{1}{3}t$, $30-t$

$\therefore A=\frac{1}{2}(6+\frac{1}{3}t)(30-t)$

$\therefore A=\frac{1}{2}\left[180-6t+10t-\frac{1}{3}t^2\right]=90-3t+5t-\frac{1}{6}t^2$



$\therefore A=90+2t-\frac{1}{6}t^2$ $\quad \therefore \frac{dA}{dt}=2-\frac{1}{3}t$

at t = 3 $\quad \therefore \frac{dA}{dt}=2-\frac{1}{3}(3)=1\ cm^2/sec$ $\quad$ Let $\frac{dA}{dt}=0$ $\quad \therefore 2-\frac{1}{3}t=0$ $\quad \therefore 2=\frac{1}{3}t$ $\quad \therefore t=6$ minute

4-a] Determine the increasing and decreasing intervals to the function f where

$f(x)=x+2\sin x$, $0<x<2\pi$

The Solution

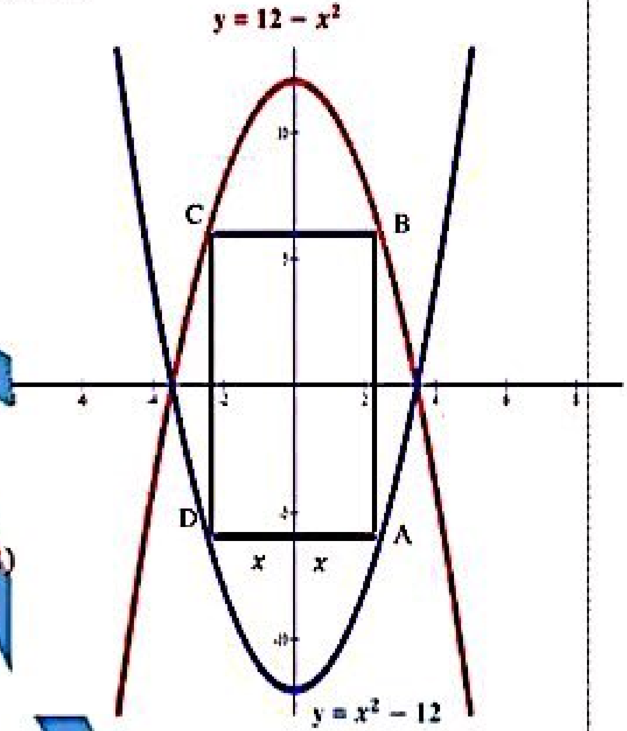
$f'(x)=1+2\cos x$, Let $f'(x)=0$, $\therefore 1+2\cos x=0$ $\quad \therefore 2\cos x=-1$ $\quad \therefore \cos x=\frac{-1}{2}$

$\therefore m(\angle x)=120$ $\quad \therefore m(\angle x)=180+60=240°$

The function is decreasing in $]\frac{2}{3}\pi,\frac{4}{3}\pi[$, the function is increasing in $]0,\frac{2}{3}[$ and $]\frac{4}{3}\pi,2\pi[$

4-b] A rectangle is drawn such that two adjacent vertices of the rectangle lie on the curve $y = x^2 - 12$ and the other two vertices lie on the curve $y = 12 - x^2$, find the maximum area of this rectangle.

The Solution



Let AD = $2x$ unit length

From the geometric graph

$A(x\,,\ x^2 - 12)$ & $B(x\,,\ 12 - x^2)$

$AB = 12 - x^2 - x^2 + 12 = 24 - 2x^2$

The area of the rectangle = AD × AB = $2x(24 - 2x^2)$

$f(x) = 48x - 4x^3$

$\therefore f'(x) = 48 - 12x^2 = -12(x^2 - 4) = -12(x-2)(x+2)$

$f'(x) = 0$ ***when*** $x = -2$ ***refused*** $\therefore x = 2$

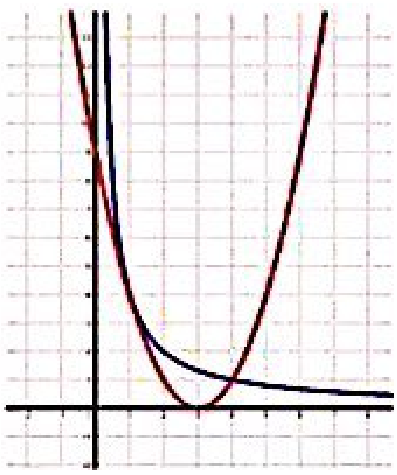
$f''(x) = -24x \quad \because f''(2) = -48 < 0$

$\therefore$ ***at*** $x = 2$ ***the area is maximum***

$\therefore$ ***The area of the rectangle*** $= 48 \times 2 - 4 \times 8 = 64$ ***square unit .***

5-a] Find the volume of the solid generated by revolving the region bounded by the two curves $y = \frac{4}{x}$ and $y = (x-3)^2$ a complete revolution about x –axis .

The Solution



Let $y_1 = \frac{1}{x}$, $y_2 = (x-3)^2$ to find the point of intersection

$\therefore (x-3)^2 = \frac{4}{x} \qquad \therefore x[x^2 - 6x + 9] = 4$

$\therefore x^3 - 6x^2 + 9x - 4 = 0 \qquad \therefore (x^3 - 1) - (6x^2 - 9x + 3) = 0$

$\therefore (x-1)[x^2 + 2x + 1] - 3[2x^2 - 3x + 1] = 0$

$\therefore (x-1)[x^2 + 2x + 1] - 3(2x-1)(x-1) = 0$

$\therefore (x-1)[x^2 + 2x + 1 - 3(2x-1)] = 0$

$\therefore (x-1)[x^2 + 2x + 1 - 6x + 3] = 0 \qquad \therefore (x-1)[x^2 - 5x + 4] = 0$

$\therefore (x-1)(x-1)(x-4)=0 \qquad \therefore x = 1 \quad \therefore x = 4$

∴ Remark the point of intersection from the graph , ∵ $y_1 \geq y_2$ for every $x \in [1\ ,\ 4]$

$\therefore v = \pi \int_1^4 (y_1^2 - y_2^2)\, dx = \pi \int_1^4 \left[\frac{16}{x^2} - (x-3)^4\right] dx = \pi \left[\frac{-16}{x} - \frac{1}{5}(x-3)^5\right]_1^4$

$= \pi[-\frac{16}{4} - \frac{1}{5}(1) - \left(-16 - \frac{1}{5}(-2)^5\right)] = \frac{27}{5}\pi$ cubic meter

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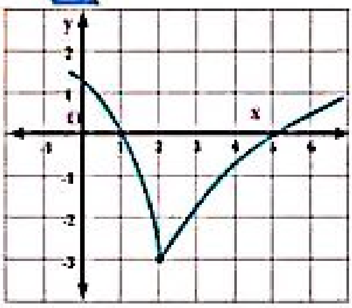
5-b] Sketch the curve of the function f which satisfies the following properties:

$f(1) = f(5) = 0\ ,\ f(2) = -3$, $f''(x) < 0$ for each $x \neq 0$

$f'(x) < 0$ for each $x < 2$, $f'(x) > 0$ for each $x > 2$

The Solution



∵ $f(1) = f(5) = 0\ ,\ f(2) = -3$, ∴ The curve passes through

$(1, 0), (5, 0), (2, -3)$ ∵ $f''(x) < 0$ for each $x \neq 2$

∴ The curve concave up in the interval $]-\infty, 2[\ ,\ ]2, \infty[$

∵ $f'(x) < 0$ for each $x < 2$, $f'(x) > 0$ for each $x > 2$

∴ $(2, -3)$ is local minimum point

$f(x)$ is decreasing in $]-\infty, 2[$, $f(x)$ is increasing in $]2, \infty[$

Test 2

1] Choose the correct answer :

1– The equation of the tangent to the curve of the function f where $f(x) = e^{2x+1}$ at point $(\frac{-1}{2}, 1)$ is :

a] $2y = x + 1$ b] $y = 2x + 2$

c] $y = 2x - 3$ d] $2y = 3x + 1$

2– If $y = 4n^3 + 4$, $z = 3n^2 - 2$, then the rate of change of z with respect to y– equals:

a] 2 n **b] 2** **c] $\frac{1}{2n}$** **d] 4**

3 - The maximum value of the expression $8x - x^2$ where $x \in R$ is

a] 8 b] 16 c] 32 d] 64

4 - If the slope of the tangent to the curve of the function f at any point on it equals $\frac{1}{x-2}$ and the curve passes through point $(3, 0)$, then $f(e^2 + 2)$ equals

a] 2 b] 3 c] Ln 2 d] Ln 3

5 – If f is a continuous function on R , $\int_1^2 f(x)dx = 9$ & $\int_6^2 f(x)dx = -7$ then $\int_1^6 f(x)dx$ equals:

a] 2 b] 8 c]16 d]−63

6– The volume of the solid generated by revolving the region bounded by the curve $y = \sqrt{x+1}$ and the straight lines $y = 0$, $x = -1$ and $x = 1$ equals

a] π b] $\frac{3\pi}{2}$ c] 2π d] $\frac{5\pi}{2}$

The Solution

(1) $\because f(x) = e^{2x+1}$ $\therefore f'(x) = 2e^{2x+1}$ $\therefore m\left[\text{the slope at } \left(\frac{-1}{2}, 1\right)\right] = 2$

The equation of the tangent $\frac{y-y_1}{x-x_1} = m$ $\therefore \frac{y-1}{x-\frac{-1}{2}} = 2$ $\therefore y - 1 = 2\left(x + \frac{1}{2}\right)$

$\therefore y - 1 = 2x + 1$ $\therefore y = 2x + 2$

(2) $\frac{dy}{dt} = 12n^2$, $\frac{dx}{dt} = 6n$, $\frac{dx}{dy} = \frac{dx}{dt} \div \frac{dy}{dt} = \frac{6n}{12n^2} = \frac{1}{2n}$

(3) Let $f(x) = 8x - x^2$, $\therefore f'(x) = 8 - 2x$, $f''(x) = -2$, Let $f'(x)$, $\therefore x = 4$, $\therefore f''(4) = -2 < 0$

$\therefore$ at $x = 4$ the expression has the maximum value $\therefore f(4) = 8(4) - (4)^2 = 32 - 16 = 16$

(4) $f'(x) = \frac{1}{x-2}$ $\therefore f(x) = \int \frac{dx}{x-2}$ $\therefore f(x) = Ln\,|x-2| + c$ $\because$ the curve passes through (3 , 0)

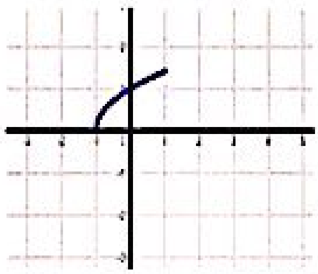
$\therefore 0 = Ln\,|1| + c$ $\therefore c = 0$ $\therefore f(x) = Ln\,|x-2|$ $\therefore f(e^2+2) = Ln\,|e^2+2-2| = Ln\,|e^2| = 2Ln\,e = 2$

(5) $\int_1^6 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^6 f(x)\,dx = 9 - \int_6^2 f(x)\,dx = 9 - -7 = 16$

(6) $v = \pi\int_{-1}^{1} y^2\,dx = \int_{-1}^{1} \pi(x+1)\,dx$

$= \pi\left[\frac{1}{2}x^2 + x\right]_{-1}^{1} = \pi\left[\left(\frac{1}{2}+1\right) - \left(\frac{1}{2}-1\right)\right]$

$= 2\pi$ unit volume



2–a] $\int x(2x-1)^3\,dx$, $\int x\,e^{-2x}\,dx$

The Solution

$\int x(2x-1)^3\,dx$, Let $y = 2x - 1$ $\therefore dy = 2\,dx$ $\therefore dx = \frac{1}{2}dy$, $y + 1 = 2x$ $\therefore x = \frac{1}{2}(y+1)$

$\therefore I = \int \frac{1}{2}(y+1) \times y^3 \times \frac{1}{2}dy = \int \frac{1}{4}y^3(y+1)\,dy = \int \frac{1}{4}y^4\,dy + \int \frac{1}{4}y^3 dy = \frac{1}{20}y^5 + \frac{1}{16}y^4 + c =$
$\frac{1}{20}(2x-1)^5 + \frac{1}{16}(2x-1)^4 + c$

Let $u = x$, $e^{-2x}dx = dv$ $\therefore dv = dx$, $\frac{-1}{2}e^{-2x} = v$

$\therefore I = uv - \int v\,du = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c$

2-b] Find the rate of change for $\sqrt{16+x^2}$ with respect to $\frac{x}{x-2}$ when $x = -3$

The Solution

Let $y = \sqrt{16+x^2} = (16+x^2)^{\frac{1}{2}}$ $\therefore \frac{dy}{dx} = \frac{1}{2}(16+x^2)^{-\frac{1}{2}}(2x) = \frac{x}{(16+x^2)^{\frac{1}{2}}} = \frac{x}{\sqrt{16+x^2}}$

Let $z = \frac{x}{x-2}$ $\therefore \frac{dz}{dx} = \frac{1(x-2)-1(x)}{(x-2)^2} = \frac{x-2-x}{(x-2)^2} = \frac{-2}{(x-2)^2}$

$\therefore \frac{dy}{dz} = \frac{dy}{dx} \div \frac{dz}{dx} = \frac{x}{\sqrt{16+x^2}} \times \frac{(x-2)^2}{-2}$ at $x = -3$ $\therefore \frac{dy}{dz} = -\frac{3}{5} \times \frac{25}{-2} = \frac{15}{2}$

3-a] If $x\ \mathrm{Cos}\,y + y\ \mathrm{Cos}\,x = 1$, find $\frac{dy}{dx}$

The Solution

By derivative both sides by x $\therefore \mathrm{Cos}\,y + x\left(-\mathrm{Sin}\,y\frac{dy}{dx}\right) + \frac{dy}{dx}(\mathrm{Cos}\,x) + y(-\mathrm{Sin}\,x) = 0$

$\therefore \mathrm{Cos}\,y - y\,\mathrm{Sin}\,x = x\,\mathrm{Sin}\,y\frac{dy}{dx} - \mathrm{Cos}\,x\frac{dy}{dx}$ $\therefore \mathrm{Cos}\,y - y\,\mathrm{Sin}\,x = \frac{dy}{dx}[x\,\mathrm{Sin}\,y - \mathrm{Cos}\,x]$

$\therefore \frac{dy}{dx} = \frac{\mathrm{Cos}\,y - y\,\mathrm{Sin}\,x}{x\ \mathrm{Sin}\,y - \mathrm{Cos}\,x}$

3-b] Find the absolute extrema values of the function f in the interval $[-1, 1]$ where

$f(x) = 2x^3 + 6x^2 + 5$

The Solution

$\because f(x) = 2x^3 + 6x^2 + 5$ $\therefore f'(x) = 6x^2 + 12x = 6x(x+2)$ Let $f'(x) = 0$

$\therefore x = 0$ or $x = -2$ but $-2 \notin [-1, 1]$ (refused) $\therefore f(0) = 5$ absolute minimum

$f(-1) = 9$, $f(1) = 13$ absolute maximum

4-a] If $f(x) = \begin{cases} 2x - x^2 & x \ge 0 \\ 2x + x^2 & x < 0 \end{cases}$ find:

i) The local maximum and minimum values of function f

ii) $\int_{-1}^{3} f(x)\,dx$

The Solution

$\because f(0^+) = f(0^-) = f(0) = 0$ $\therefore f(x)$ is a function its domain $= \mathbb{R}$ and continuous at $x = 0$

$\therefore$ The function is continuous on $\mathbb{R}$, $f'(0^+) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{2(x+h)-(x+h)^2-[2x-x^2]}{h}$ at $x = 0$

$\therefore f`(0^+) = \lim_{h\to0^+}\frac{2h-h^2}{h} = \lim_{h\to0^+}\frac{h(2-h)}{h} = \lim_{h\to0^+}\frac{h(2-h)}{h} = \lim_{h\to0^+}(2-h) = 2$

$\therefore f`(0^-) = \lim_{h\to0^-}\frac{f(x+h)-f(x)}{h} = \lim_{h\to0^-}\frac{2(x+h)+(x+h)^2-[2x+x^2]}{h}$ at $x = 0$

$\therefore f`(0^-) = \lim_{h\to0}\frac{2h+h^2}{h} = \lim_{h\to0}\frac{h(2+h)}{h} = \lim_{h\to0}(2+h) = 2$, $\therefore f`(0^+) = f`(0^-) = 2$

$$\therefore f`(x) = \begin{cases} 2+2x & when\ x<0 \\ 2 & when\ x=0 \\ 2-2x & when\ x>0 \end{cases}$$ , Let $f`(x) = 0$

$\therefore 2+2x = 0$, when $x<0$ $\therefore x = -1$ and $2-2x = 0$ at $x>0$ $\therefore x = 1$

$\therefore (1, 1)$ and $(-1, -1)$ are two critical points, $\therefore f(1) = 1$ Local maximum point
$f(-1) = -1$ Local minimum point

$$\therefore \int_{-1}^{3} f(x) = \int_{-1}^{0}(2x+x^2)\,dx + \int_{0}^{3}(2x-x^2)\,dx$$
$$= \left[x^2+\frac{1}{3}x^3\right]_{-1}^{0} + \left[x^2-\frac{1}{3}x^3\right]_{0}^{3} = \left(0-1+\frac{1}{3}\right)+(9-9-0) = \frac{-2}{3}$$

4-b] The volume of a cube increases regularly such that it keeps its shape at a rate of $27\ cm^3/min$, **find the increase of the area** of its faces at the moment which its edge length is 3 cm.

The Solution

Let the length of the side of the cube is L unit length

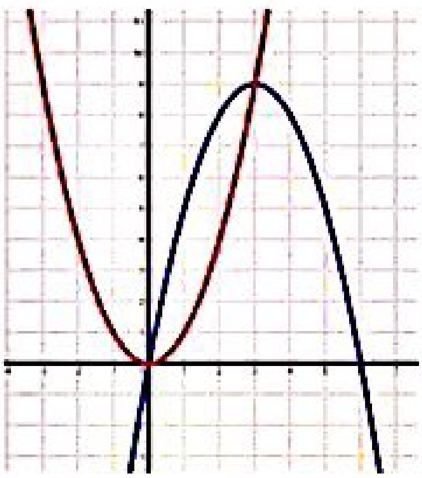
$\therefore$ The volume of the cube $v = L^3$ unit volume

$\therefore \frac{dv}{dt} = 3L^2\frac{dL}{dt}$ $\quad\therefore 27 = 3\times9\times\frac{dL}{dt}$ $\quad\therefore \frac{dL}{dt} = \frac{27}{27} = 1$ cm/min.

$\because$ The total surface area of the cube $A = 6L^2$, $\therefore \frac{dA}{dt} = 12L\times\frac{dL}{dt} = 12\times3\times1 = 36\ cm^2/min.$

5-a] Find the area of the region bounded by the two curves $y = x^2$ and $y = 6x - x^2$ in Square units

The Solution



Let $y_1 = x^2$, $y_2 = 6x - x^2$

to find the point of intersection

$\therefore y_1 = y_2$ $\therefore x^2 = 6x - x^2$ $\quad\therefore 2x^2 - 6x = 0$

$\therefore 2x(x-3) = 0$ $\quad\therefore x = 0$ or $x = 3$

Remark :

The point of intersection from the opposite graph

$\because y_2 \geq y_1$ for each $x \in [0, 3]$

$\therefore A = \int_0^3 (y_2 - y_1)\,dx = \int_0^3 (6x - x^2 - x^2)\,dx$

$= \int_0^3 (6x - 2x^2)\,dx = \left[3x^2 - \frac{2}{3}x^3\right]_0^3 = (27 - 18) - 0 = 9$ square unit

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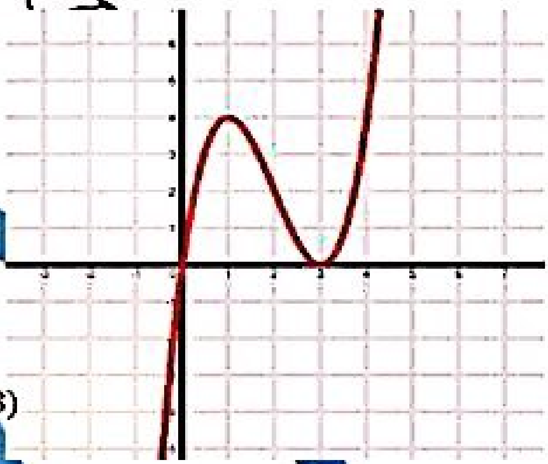
5−b] If the function f where $f(x) = x^3 + a\,x^2 + b\,x$ has an inflection point at (2 , 2) , find the two values of the two constants a and b , then sketch the curve of the function.

The Solution



$f(x) = x^3 + ax^2 + bx$ $\qquad \therefore f'(x) = 3x^2 + 2ax + b$

$\therefore f''(x) = 6x + 2a$ $\because$ (2 , 2) is an inflection point

$\therefore f''(2) = 0 \;\therefore 12 + 2a = 0 \qquad \therefore a = -6$

$\because$ The curve passes through (2 , 2) $\therefore f(2) = 2$

$\therefore 8 - 24 + 2b = 2 \quad \therefore b = 9 \quad \therefore f(x) = x^3 - 6x^2 + 9x$

$\therefore f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$

$f''(x) = 6x - 12 = 6(x - 2)$ when $f'(x) = 0$, $\therefore x = 3 , x = 1$

$\therefore$ The function has critical point at $x = 3 , x = 1$

Because $f'(x)$ change its sign before and after $x = 3 , x = 1$

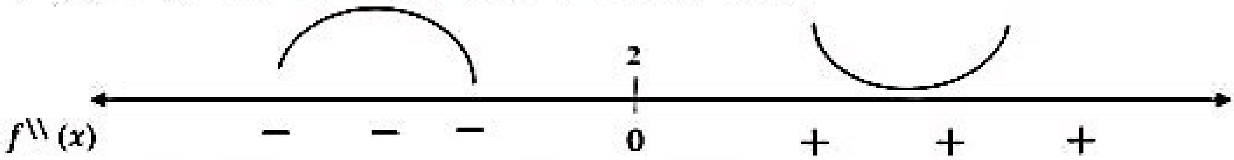
The function is increasing in the interval $]3 , \infty[$ and $]-\infty , 1[$

& the function is decreceing at $]1 , 3[$ when $f''(x) = 0$, $\therefore x = 2$, $\because f''(3) = 6 > 0$

$\therefore f(3) = 0$ local minimum value , $f''(1) = -6 < 0$, $\therefore f(1) = 4$ local maximum value

$\because f''(x) < 0$ in $]-\infty , 2[$ $\quad \therefore$ the curve is concave up in this interval

$\because f''(x) > 0$ in $]2 , \infty[$ $\quad \therefore$ The curve is concave down



$\because$ The point (2 , 2) is an inflection point (3 , 0) , (0 , 0) are the point of intersection with the x −axis (4 , 4) , (−1 , −16)

Test 3

Choose the correct answer :

1– The slope of the tangent to the curve of the circle $x^2 + y^2 = 25$ when $x = 3$ equals

a] $\frac{-4}{3}$ b] $\frac{-3}{4}$ C] $\frac{5}{12}$ d] $\frac{4}{3}$

2– If $f(x) = \frac{x}{x-2}$, then $f'''(3)$ equals

a] −36 b] −12 C] 6 d] 4

3– If $\frac{dy}{dx} = \text{Csc}^2$, y = 2 and $x = \frac{\pi}{4}$, then y equals

a] $-(2 + \text{Cot}\, x)$ b] $-(3 + \text{Cot}\, x)$ c] $2 - \text{Cot}\, x$ d] $3 - \text{Cot}\, x$

4– If $\int_2^4 f(x)dx = 7$, $\int_4^2 g(x)dx = 2$, *then* $\int_2^4 [2f(x) - 3g(x) - 5]\, dx$ *equals*

a] −18 b] −8 c] 10 d] 14

5– The area of the region bounded by the straight lines $y = 2x - 3$, $y = x + 1$, $x = 2$ equals

a] 2 b] 3 c] $\frac{9}{2}$ d] 6

6 – The volume of the solid generated by revolving the region bounded by the two curves $y = \tan\theta$, and $y = \text{Sec}\,\theta$ and the two straight lines $x = \frac{\pi}{6}$, $x = \frac{\pi}{3}$ a complete revolution about x –axis approximated in cubic units squares :

a] $\frac{\pi}{6}$ b] $\frac{\pi}{3}$ c] $\frac{2\pi}{5}$ d] 2π

The Solution

(1) $\because x^2 + y^2 = 25$ by derivative each side with respect to x $\therefore 2x + 2y\frac{dy}{dx} = 0$

$\therefore 2y\frac{dy}{dx} = -2x$ $\therefore \frac{dy}{dx} = \frac{-x}{y}$ at $x = 3$ $\therefore y = \pm 4$ $\therefore$ the slope of the tangent $= \frac{dy}{dx} = \frac{-3}{4}$

(2) $\because f(x) = x(x-2)^{-1}$ $\therefore f'(x) = (x-2)^{-1} - x(x-2)^{-2}$

$\therefore f''(x) = -(x-2)^{-2} - (x-2)^{-2} + 2x(x-2)^{-3}$ $\therefore f''(x) = -2(x-2)^{-2} + 2x(x-2)^{-3}$

$f'''(x) = 4(x-2)^{-3} + 2(x-2)^{-3} - 6x(x-2)^{-4}$

$\therefore f'''(3) = 4(3-2)^{-2} + 2(3-2)^{-3} - 6(3)(3-2)^{-4} = 4 + 2 - 18 = -12$

(3) $\frac{dy}{dx} = Cosec^2\, x$, $\therefore y = \int Cosec^2\, x\; dx = -\text{Cot}\, x + c$, $\because y = 2$ when $x = \frac{\pi}{4}$, $\therefore 2 = -1 + c$

$\therefore c = 3$, $\therefore y = -\text{Cot}\, x + 3$

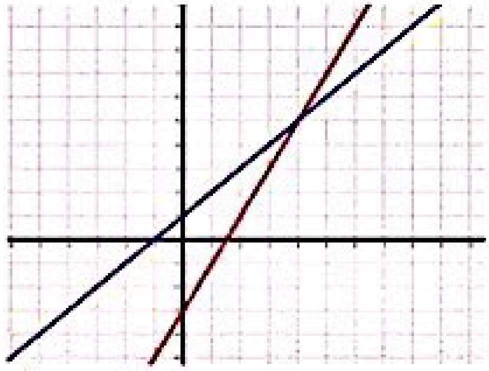
(4) $\int_2^4 g(x)\; dx = -2$

$\therefore \int_2^4 [2f(x) - 3g(x) - 5]\; dx = 2\int_2^4 f(x)\; dx - 3\int_2^4 g(x)\; dx - \int_2^4 5\, dx = 2 \times 7 - 3 \times -2 - [5x]_2^4$

$= 14 + 6 - [20 - 10] = 10$.

(5) Let $y_1 = 2x - 3$, $y_2 = x + 1$

to find the point of intersection



Let $y_1 = y_2$ $\therefore 2x - 3 = x + 1$

$\therefore x = 4$ $\because y_2 \geq y_1$ for each $x \in [2,4]$

$\therefore A = \int_2^4 (y_2 - y_1)\, dx = \int_2^4 (x + 1 - 2x + 3)\, dx = \int_2^4 (4 - x)\, dx$

$= \left[4x - \frac{1}{2}x^2\right]_2^4 = (16 - 8) - (8 - 2) = 2$ square unit

(6) Let $y_1 = \tan\theta$, $y_2 = Sec\,\theta$, $\because y_2 \geq y_1$ for each $x \in \left[\frac{\pi}{6} , \frac{\pi}{3}\right]$

$\therefore v = \pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (y_2^2 - y_1^2)\, dx = \pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (Sec^2\,\theta - tan^2\,\theta)\, d\theta = \pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1\, d\theta = [\pi\,\theta]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \pi[\frac{\pi}{3} - \frac{\pi}{6}] = \frac{1}{6}\pi^2$ unit of volume

---

2-a] Find the derivative of y with respect to x where : $y = x^2\ Ln\,x$

The Solution

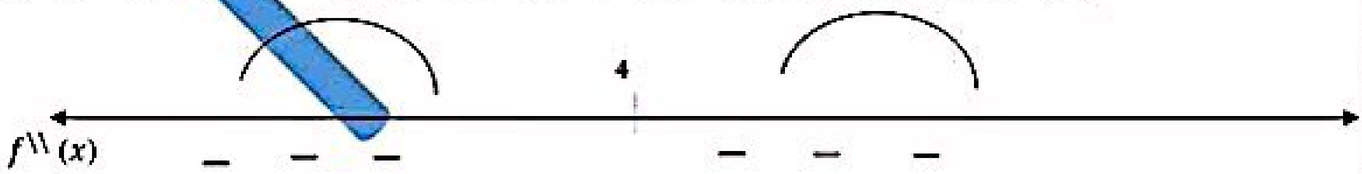
$\frac{dy}{dx} = x^2\left(\frac{1}{x}\right) + (Ln\,x)(2x) = x + 2x\,Ln\,x = x[1 + 2\,Ln\,x] = x[1 + Ln\,x^2]$

2-b] If $f(x) = \sqrt[3]{(x-4)^2}$ find the convexity intervals upwards and downwards and the inflection points (if existed) to the curve of the function f .

The Solution

$f(x) = (x - 4)^{\frac{2}{3}}$ $\therefore f\,'(x) = \frac{2}{3}(x - 4)^{-\frac{1}{3}}$ $\therefore f\,''(x) = \frac{-2}{9}(x - 4)^{\frac{-4}{3}} = \frac{-2}{9\sqrt[3]{(x-4)^4}}$

$f\,''(x) < 0$ when $4 < x$ $\therefore f''(x) < 0$ when $4 < x$, $f''(x)$ for each $x \in R - \{4\}$



Remark : $f''(x)$ is not exist when $x = 4$, the curve is convex up in the two interval $]-\infty, 4[$ and $]4, \infty[$, and there is no inflection point because $f''(x)$ does not change its sign before and after $x = 4$

3–a] Find $\int x(x-5)^3\,dx$, $\int 4x\,e^{2x}\,dx$

The Solution

Let $y = x-5$ $\therefore dy = dx$

$\therefore I = \int y^3(y+5)\,dy = \int y^4 + 5y^3\,dy = \frac{1}{5}y^5 + \frac{5}{4}y^4 + c = \frac{1}{5}(x-5)^5 + \frac{5}{4}(x-5)^4 + C$

Another solution :

$\int x(x-5)^3\,dx = \int(x-5+5)(x-5)^3\,dx = \int(x-5)^4\,dx + 5\int(x-5)^3\,dx$

$= \frac{1}{5}(x-5)^5 + \frac{5}{4}(x-5)^4 + c = \frac{1}{20}(x-5)^4[4x-20+25] + C = \frac{1}{20}(x-5)^4(4x+5) + C$

$\therefore I = \int 4xe^{2x}\,dx$ Let $u = 4x$, $e^{2x}\,dx = dv$ $\therefore du = 4\,dx$, $\frac{1}{2}e^{2x} = v$

$\therefore I = uv - \int v\,du = (4x)\left(\frac{1}{2}e^{2x}\right) - \int(4)\left(\frac{1}{2}e^{2x}\,dx\right) = 2xe^{2x} - 2\int e^{2x}\,dx$

$= 2xe^{2x} - e^{2x} + c = e^{2x}(2x-1) + c$

3–b] Find the absolute maximum values of the function f where $f(x) = x^4 - 4x^3$ on the interval $[0,4]$

The Solution

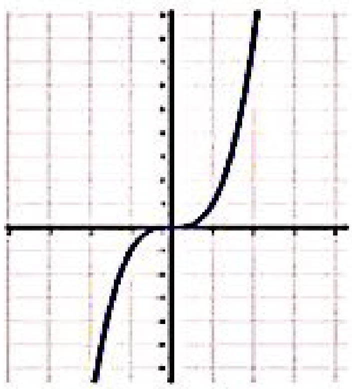
$f(x) = x^4 - 4x^3$ $\therefore f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

Let $f'(x) = 0$ $\therefore x = 0 \in [0,4]$, $x = 3 \in [0,4]$

$\because f(0) = 0$ absolute maximum, $f(3) = -27$ absolute maximum , $f(4) = 0$

4–a] The volume of a solid of revolution generated by revolving the region bounded by the curve $y = x^3$ and the two straight lines $x = 0$ and $y = 1$ a complete revolution about x –axis is equal to the volume of a cylinder–like wire whose length is 42 units. What is the radius length of that wire?

The Solution



Let $y_1 = x^3$, $y_2 = 1$

to find the point of intersection

Let $y_1 = y_2$ $\therefore x^3 = 1$ $\therefore x = 1$

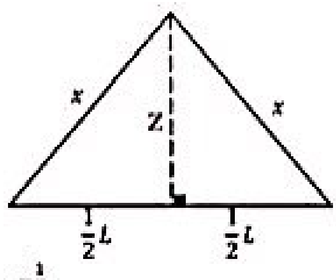
$\because y_2 \geq y_1$ for each $x \in [0,1]$ $\therefore$ The rotation around x –axis

$v = \pi\int_0^1 (y_2^2 - y_1^2)dx = \pi\int_0^1 (1 - x^6)\,dx$

$= \pi\left[x - \frac{1}{7}x^7\right]_0^1 = \pi\left[1 - \frac{1}{7}\right] = \frac{6}{7}\pi$ cubic unit.

4-b] The two equal legs of the isosceles triangle with a constant base whose length is L cm/min decrease at a rate of 3 cm/min. what is the rate of the decrease in the area when the triangle becomes an equilateral triangle?

The Solution



Let the length of each legs L cm , $\frac{dx}{dt} = -3$ cm/min.

from the graph $z^2 = x^2 - \frac{1}{4}L^2 \quad \therefore Z = \left(x^2 - \frac{1}{4}L^2\right)^{\frac{1}{2}}$

$\therefore$ The area of the triangle $= \frac{1}{2}LZ = \frac{1}{2}L\left(x^2 - \frac{1}{4}L^2\right)^{\frac{1}{2}}$

$\therefore \frac{dA}{dt} = \frac{1}{4}L\left(x^2 - \frac{1}{4}L^2\right)^{-\frac{1}{2}}\left(2x\frac{dx}{dt}\right) = \frac{1}{2}Lx\left(x^2 - \frac{1}{4}L^2\right)^{-\frac{1}{2}} \times -3 = \frac{-3}{2}Lx\left(x^2 - \frac{1}{4}L^2\right)^{-\frac{1}{2}}$

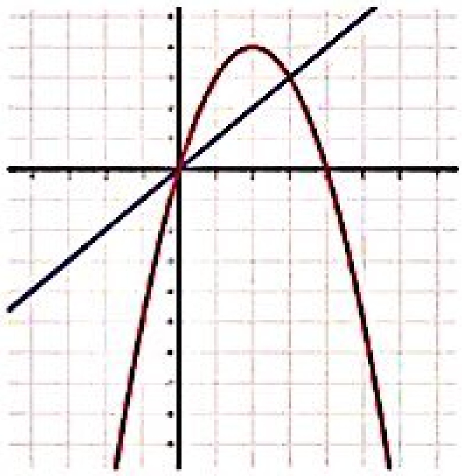
To become the triangle is equilateral $\therefore x = L$

$\therefore \frac{dA}{dt} = \frac{-3}{2}x^2\left(x^2 - \frac{1}{4}x^2\right)^{-\frac{1}{2}} = \frac{-3}{2}x^2\left(\frac{3}{4}x^2\right)^{-\frac{1}{2}} = \frac{\frac{-3}{2}x^2}{\frac{1}{2}\sqrt{3}x} \times \frac{\sqrt{3}}{\sqrt{3}} = -\sqrt{3}x\ cm^2$/min.

$\therefore$ The area is decreased by the rate $\sqrt{3}x\ cm^2$/min.

5-a] Find the area of the region bounded by the two curves $x - y = 0$, $y = 4x - x^2$

The Solution



Let $y_1 = x$, $y_2 = 4x - x^2$

To find the point of intersection

Let $y_1 = y_2 \quad \therefore 4x - x^2 = x$

$\therefore x^2 - 3x = 0 \qquad \therefore x(x-3) = 0$

$\therefore x = 0$ or $x = 3$

$\because y_2 \geq y_1$ for each $x \in [0, 3]$

$\therefore A = \int_0^3 (y_2 - y_1)\, dx = \int_0^3 (4x - x^2 - x)\, dx = \int_0^3 (3x - x^2)\, dx$

$= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3 = \left(\frac{27}{2} - 9\right) - 0 = \frac{9}{2}$ square unit.

5-b] Sketch the curve of the continuous function f which has the following properties:

1- $f(0) = 3$

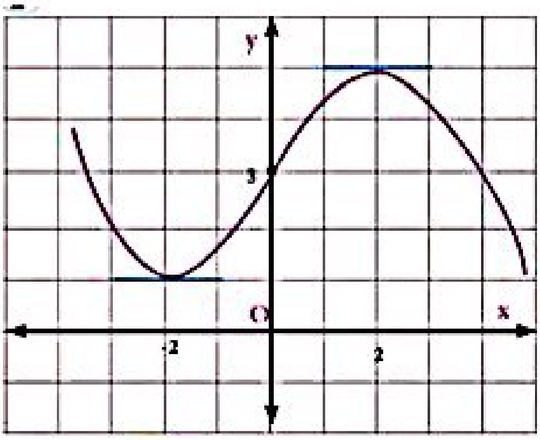
2- $f'(2) = f'(-2) = 0$

3- $f'(x) > 0$ when $-2 < x < 2$

4- $f''(x) < 0$ when $x > 0$, $f''(x) > 0$ when $x < 0$.

The Solution

$\because f(0) = 3$ $\therefore$ The curve passes through $(0, 3)$



$\because f'(2) = f'(-2) = 0$

$\therefore$ The curve have critical point at $x = 2, x = -2$

$\because f'(x) < 0$ when $-2 < x < 2$

$\therefore f(x)$ increasing in $]-2, 2[$, $\because f''(x) < 0$ when $x > 0$

$\therefore$ The curve concave up in the $]0, \infty[$

$\because f''(x) > 0$ when $x < 0$

$\therefore$ The curve is concave down in $]-\infty, 0[$

The point $(0, 3)$ is an inflection point.

Test 4

Choose the correct answer

1 – If $y = \frac{3x-5}{x-2}$, at $x = 1$, $\frac{d^2y}{dx^2}$ equals

a] -12 b] -6 c] 6 d] 12

2 – $\int Sec^3 x \tan x \, dx$ equals :

a] $\frac{1}{4} Sec^4 x + c$ b] $\frac{1}{3} Sec^3 x + c$ c] $\frac{1}{2} tan^2 x + c$ d] $\frac{-1}{2} tan^2 x + c$

3– The normal to circle $x^2 + y^2 = 12$ at any point in it passes thought point

a] $(2, 3)$ b] $(1, 1)$ c] $(0, 0)$ d] $(-2, -2)$

4– The curve of the function f where $f(x) = (x - 2)\, e^x$ is convex downwards on the interval :

a) $]-\infty, \infty[$ b) $]-1, 2[$ c) $]0, 2[$ d) $]0, \infty[$

5 – $\int_{-1}^{3} 3x\,|x - 4|\, dx$ equals

a] -27 b] -20 c] 20 d] 27

6– When the region bounded by the curve $x = \frac{1}{\sqrt{y}}$, $1 \le y \le 4$ and y–axis revolves a complete revolution about y–axis , then the volume of the solid generated approximated in cubic units equals:

a] $\frac{2}{3}\pi$ b] $3\sqrt{2}\,\pi$ c] $2\pi \log_e 2$ d] $\frac{2}{3}\pi$ Log 3

The Solution

(1) $y = (3x-5)(x-2)^{-1}$ $\therefore \frac{dy}{dx} = 3(x-2)^{-1} - (x-2)^{-2}(3x-5) = (x-2)^{-2}[3(x-2) - 3x + 5]$

$= (x-2)^{-2}[3x - 6 - 3x + 5] = (x-2)^{-2}[-1] = -(x-2)^{-2}$

$\frac{d^2y}{dx^2} = 2(x-2)^{-3}$ $\therefore$ $\frac{d^3y}{dx^3} = -6(x-2)^{-4} = \frac{-6}{(x-2)^4}$ at $x = 1$ $\therefore \frac{d^3y}{dx^3} = \frac{-6}{(1-2)^4} = -6$

(2) $\int Sec^3 x \tan x \, dx = \frac{1}{3} Sec^3 x + c$

(3) $\because x^2 + y^2 = 25$ $\therefore 2x + 2y\frac{dy}{dx} = 0$ $\therefore \frac{dy}{dx} = \frac{-x}{y}$

Let the point (c , d) on the circle $\therefore$ The slope of the tangent $= \frac{dy}{dx}$ [at (c , d)] $= \frac{-c}{d}$

$\therefore$ The slope of the normal $= \frac{d}{c}$ $\therefore$ The equation of the normal $\frac{y-y_1}{x-x_1} = \frac{-1}{m}$

$\therefore \frac{y-d}{x-c} = \frac{d}{c}$ $\therefore cy - cd = dx - cd$ $\therefore cy = dx$

and this is an equation passes through (0 , 0) & the equation of the diameter of the circle

$\therefore$ The normal to the circle at the point (0 , 0) .

(4) $f(x) = (x-2)e^x$ $\therefore f'(x) = e^x + (x-2)e^x = e^x[1 + 2x - 2] = e^x(x-1)$

$f''(x) = e^x(x-1) + e^x = e^x[x - 1 + 1] = xe^x$

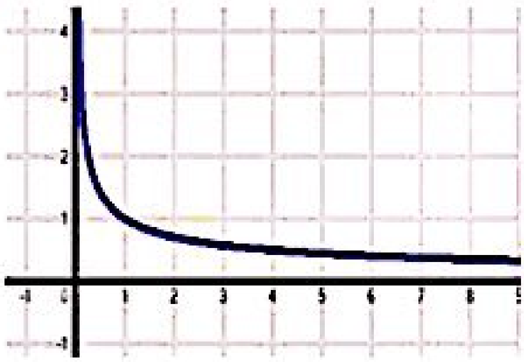
$\because f''(x) = 0$ when $x = 0$, $\because f''(x) > 0$ when $x > 0$

$\because$ The curve is convex down in $]0, \infty[$

(5) $\int_{-1}^{3} 3x|x-4|\,dx = \int_{-1}^{3} 3x(-x+4)\,dx = \int_{-1}^{3}(-3x^2 + 12x)\,dx = [-x^3 + 6x^2]_{-1}^{3}$

$= (-27 + 54) - (1 + 6) = 20$

(6) $\because$ The rotation around y−axis



$\therefore v = \pi \int_1^4 x^2 dy = \int_1^4 \frac{1}{y} dy$

$= \pi[\ Log_e|y|\ ]_1^4 = \pi[\ Log_e 4 - Log_e 1\ ]$

$= \pi\ Log_e 4 = \pi\ Log_e 2^2$

$= 2\pi\ Log_e 2$ cubic unit .

2−a] Find : $\int (3x^2 - 4e^2 x)\,dx$ $\int \frac{x-1}{\sqrt{x+3}}\,dx$

The Solution

$\int (3x^2 - 4e^{2x})\,dx = x^3 - 2e^{2x} + C$

$\int \frac{x-1}{\sqrt{x+3}}\,dx$, Let $y = x + 3$ $\therefore dy = dx$ $\therefore I = \int \frac{y-3-1}{\sqrt{y}}\,dy = \int \frac{y-4}{y^{\frac{1}{2}}}\,dy = \int y^{\frac{1}{2}} - 4y^{-\frac{1}{2}}\,dy$

$= \frac{2}{3}y^{\frac{3}{2}} - 8y^{\frac{1}{2}} + c = \frac{2}{3}(x+3)^{\frac{3}{2}} - 8(x+3)^{\frac{1}{2}} + c = \frac{2}{3}(x+3)^{\frac{1}{2}}[x + 3 - 12] + c = \frac{2}{3}(x+3)^{\frac{1}{2}}(x-9) + C$.

2-b] If $\text{Sin } y + \text{Cos } 2x = 0$, prove that : $\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 \tan y = 4 \text{ Cos } 2x \text{ Sec } y$

The Solution

$\because \text{Sin } y + \text{Cos } 2x = 0$ **derivative with respect to x** $\therefore \text{Cos } y \frac{dy}{dx} - 2 \text{ Sin } 2x = 0$

derivative with respect to x $\therefore -\text{Sin } y \frac{dy}{dx} \cdot \frac{dy}{dx} + \text{Cos } y \frac{d^2y}{dx^2} - 4 \text{ Cos } 2x = 0$

$\therefore -\text{Sin } y \left(\frac{dy}{dx}\right)^2 + \text{Cos } y \frac{d^2y}{dx^2} = 4 \text{ Cos } 2x$ divide by $\text{Cos } y$ in both sides

$\therefore \frac{-\text{Sin } y}{\text{Cos } y}\left(\frac{dy}{dx}\right)^2 + \frac{d^2y}{dx^2} = \frac{4 \text{ Cos } 2x}{\text{Cos } y}$ $\therefore \frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 \tan y = 4 \text{ Cos } 2x \text{ Sec } y$ $\therefore$ L.H.S. = R.H.S.

3-a] If $\int_1^4 f(x)\,dx = 7$, $\int_4^1 g(x)\,dx = 3$, calculate the value of

$\int_1^4 [f(x) + 2g(x) - 4]\,dx$

The Solution

$\int_1^4 f(x)\,dx + 2\int_1^4 g(x)\,dx - \int_1^4 4\,dx = \int_1^4 f(x)\,dx - 2\int_4^1 g(x)\,dx - 4\int_1^4 dx = 7 - 2(3) - [4x]_1^4$

$= 7 - 6 - [16 - 4] = 1 - 12 = -11$

3-b] If the curve of the function f where $f(x) = a\,x^3 + b\,x^2 + c\,x + d$ has a local maximum value at $(2, 4)$ and an inflection point at $(1, 2)$, find the equation of the curve.

The Solution

$f(x) = ax^3 + bx^2 + cx + d$ $\therefore f`(x) = 3ax^2 + 2bx + c$ $\therefore f``(x) = 6ax + 2b$

$\because (1, 2)$ is an inflection point $\therefore f``(1) = 0$ $\therefore 6a + 2b = 0$ $\therefore b = -3a$ ------ (1)

$\because (2, 4)$ local maximum point $\therefore f`(2) = 0$ $\therefore 12a + 4b + c = 0$ by substitution in (1)

$\therefore 12a + 4(-3a) + c = 0$ $\therefore c = 0$ ------ (2) $\because$ The curve passes through $(2, 4)$

$\therefore f(2) = 4$ $\therefore 8a + 4b + 2c + d = 4$ from (1) and (2) $\therefore 8a - 12a + d = 4$

$\therefore -4a + d = 4$ ---- (3) $\therefore$ The curve passes through $(1, 2)$ $\therefore f(1) = 2$

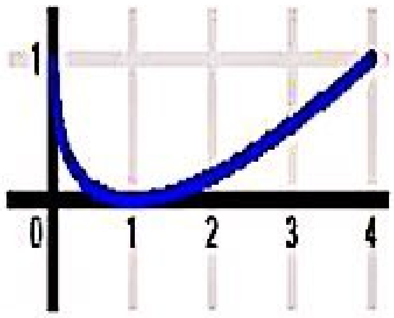
$\therefore a + b + c + d = 2$ from (1) and (2) $\therefore a - 3a + d = 2$ $\therefore -2a + d = 2$ ----- (4)

from (3) and (4) by subtraction $\therefore 2a = -2$ $\therefore a = -1$ by substitution in (1)

$\therefore b = -3a = 3$ by substitution in (4) $\therefore 2 + d = 2$ $\therefore d = 0$ $f(x) = -x^3 + 3x^2$.

4-a] Find the area of the region bounded by the curve $\sqrt{x} + \sqrt{y} = 1$ and the two straight lines $x = 0$, $y = 0$

The Solution



$\because y = 0 \quad \therefore \sqrt{x} = 1 \quad \therefore x = 1$

$\therefore$ The integration bounded by $x = 0, x = 1$

$\because \sqrt{y} = 1 - \sqrt{x}$ by squaring both sides

$\therefore y = 1 - 2\sqrt{x} + x$

$\therefore A = \int_0^1 \left(1 - 2x^{\frac{1}{2}} + x\right) dx = \left[x - \frac{4}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2\right]_0^1 = 1 - \frac{4}{3} + \frac{1}{2} = \frac{1}{6}$ square unit .

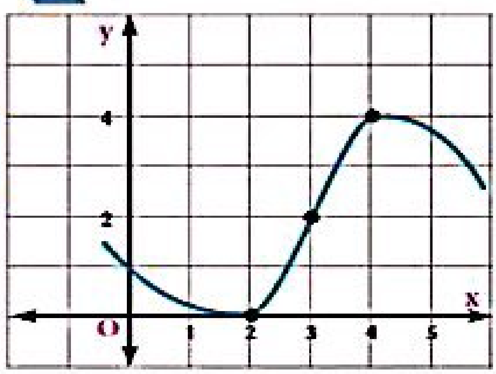
4-b] Graph the curve of the continuous function f which satisfies the following properties :

$f(4) = 2f(3) = 4$, $f(2) = 0$

$f'(x) < 0$ when $x > 4$ or $x < 2$

$f''(x) < 0$ when $x > 3$, $f''(x) > 0$ when $x < 3$

The Solution



$\because f(4) = 4 \quad \therefore$ The curve passes through $(4, 4)$

$\because 2f(3) = 4 \quad \therefore f(3) = 2$

$\therefore$ The curve passes through $(3, 2)$

$\because f(2) = 0 \quad \therefore$ The curve passes through $(2, 0)$

$\because f'(x) < 0$ when $4 < x$ or $2 > x$

$\therefore$ The function is decreasing in $]-\infty, 2[$ and $]4, \infty[$

$\because f''(x) < 0$ when $x > 3 \therefore$ The curve is convex up in $]3, \infty[$ $\because f''(x) > 0$ when $x < 3$

$\therefore$ The curve convex down in $]-\infty, 3[$ $\therefore$ The point $(3, 2)$ is an inflection point

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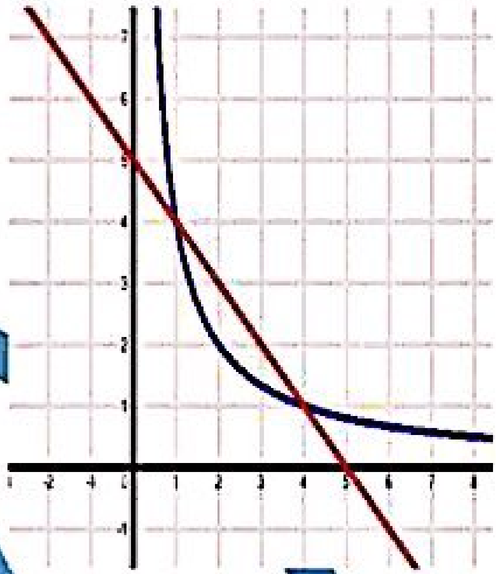
5-a] **Prove that** the volume of the solid generated by revolving the region bounded by the two curves $y = \frac{4}{x}$ and $y = 5 - x$ just one revolution about x –axis equals 9π of the cubic units

The Solution

$y_1 = \frac{4}{x}$, $y_2 = 5 - x$



to find the point of intersection

Let $y_1 = y_2$ $\therefore \frac{4}{x} = 5 - x$ $\therefore 5x - x^2 = 4$

$\therefore x^2 - 5x + 4 = 0$ $\therefore (x-1)(x-4) = 0$

$\therefore x = 1$ or $x = 4$

$\because y_2 \geq y_1$ for every $x \in [1 , 4]$

the rotation around x –axis .

$$\therefore v = \pi \int_1^4 (y_2^2 - y_1^2)dx = \pi \int_1^4 \left(25 - 10x + x^2 - \frac{16}{x^2}\right) dx$$

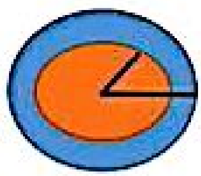
$$= \pi \left[25x - 5x^2 + \frac{1}{3}x^3 + \frac{16}{x} \right]_1^4$$

$$= \pi \left[\left(100 - 80 + \frac{64}{3} + 4\right) - \left(25 - 5 + \frac{1}{3} + 16\right)\right] = 9\pi \text{ cubic unit .}$$

5-b] If A is the area of the part bounded by two concentric circles whose radii lengths are r_1 and r_2 where $r_2 > r_1$, find the rate of change of A with respect to time at any moment at which $r_2 = 10$ cm , $r_1 = 6$cm , if known that at this moment r_1 increases at a rate of 0.3 cm/s and r_2 decreases at a rate of 0.2 cm/s.

The Solution

From the graph



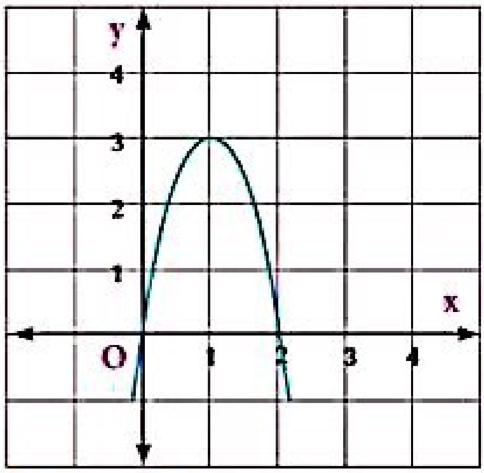
$\therefore$ The area of the part bounded by two concentric circles $S = \pi(r_2^2 - r_1^2)$

$$\therefore \frac{dA}{dt} = \pi\left[2r_2\frac{dr^2}{dt} - 2r_1\frac{dr_1}{dt}\right] = \pi[2 \times 10 \times -0.2 - 2 \times 6 \times 0.3] = -7.6\,\pi\, cm^2/\text{min.}$$

$\therefore$ The area is decreased by the rate $7.6\,\pi\ cm^2$/min.

Test 5

1] The opposite figure shows the curve $f'(x)$ of the function f where $f(x) = a\ x^3 + b\ x^2$, a , b are two constants then complete .



a] The function f is decreasing for each $x \in$ ....

b] The curve of f has critical points when $x \in$ ....

c] The curve of f is convex upwards on the interval ....

d] There is a local minimum value of the function f when $x =$ ....

e] f (1) = ...

f] The area of the region bounded by the curve of the function f and the two straight lines $x = 2$ and $y = 0$ in square units equals: ....

The Solution

$\because f(x) = ax^3 + bx^2$ $\qquad \therefore f'(x) = 3ax^2 + 2bx$

$\because$ The curve $f'(x)$ passes through the two points (1 , 3) $\therefore 3a + 2b = 3$ ....(1)

$\because$ The curve $f'(x)$ passes through the two points (2 , 0) $\therefore$ $12a + 4b = 0$ ....(2)

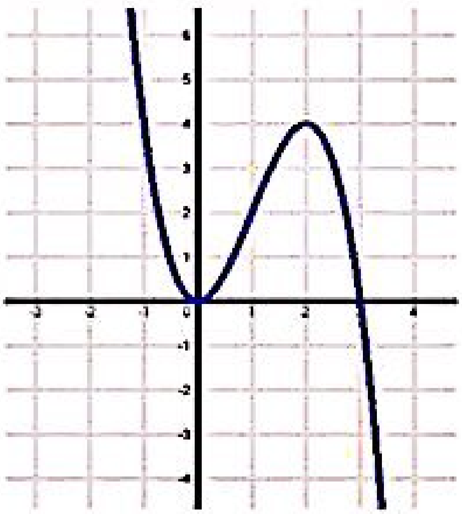
$\therefore$ by solving the two equations

$\therefore a = -1 , b = 3 \quad \therefore f(x) = -x^3 + 3x^2$

$f'(x) = -3x^2 + 6x = -3x(x-2)$

$f''(x) = -6x + 6 = -6(x-1)$

the opposite figure represent $f(x)$



(a) $\because f'(x) < 0$ when $x < 0$ or $x > 2$

$\therefore$ **The function is decreasing for each**

$x \in]-\infty , 0[$, $x \in]2 , \infty[$

(b) $\because f'(x) = 0$ when $x = 0 , x = 2$ $\therefore$ The curve critical point when $x \in \{0 , 2\}$

(c) $\because f''(x) < 0$ when $1 < x$ $\therefore$ The curve convex upwards at the interval $]1 , \infty[$

(d) $\because f'(x) < 0$ when $x < 0$, $f'(x) > 0$ when $x > 0$ $\therefore$ there exist local minimum point for the function when $x = 0$ **Remark** $f''(0) > 0$

(e) f(1) = 2

(f) The area of the region bounded by the curve and the two lines $x = 2, x = 0$

$\therefore A = \int_0^2 (-x^3 + 3x^2)\,dx = \left[-\frac{1}{4}x^4 + x^3\right]_0^2 = -4 + 8 - 0 = 4$ square unit

2–a] Find : $\int Csc^2 \frac{x+5}{2}\,dx$ $\int \frac{5x}{3x^2-1}\,dx$

The Solution

$\int Cosec^2 \frac{x+5}{2}\,dx = -2\,Cot\frac{x+5}{2} + c$

$\int \frac{5x}{3x^2-1}\,dx = \frac{5}{6}\int \frac{6x}{3x^2-1}\,dx = \frac{5}{6}\log_e|3x^2 - 1| + c$

2–b] The function f where $f(x) = x^3 - 6x^2 + 9x - 1$

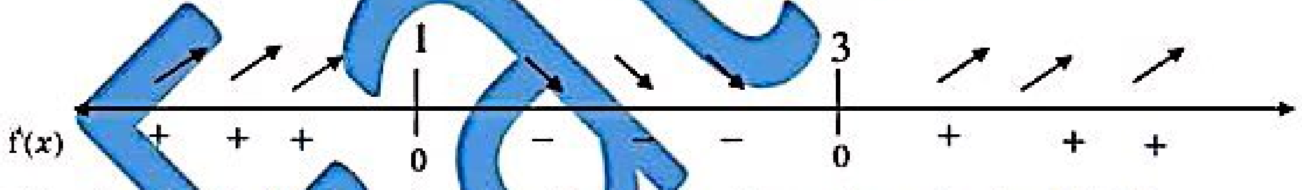
1- Determine the increasing and decreasing intervals of function f

2- Find the absolute maximum values of the function f in the interval [0 , 2]

The Solution

$f(x) = x^3 - 6x^2 + 9x - 1$ $\therefore f'(x) = 3x^2 - 12x + 9 = 3[x^2 - 4x + 3] = 3(x-1)(x-3)$

Let $f'(x) = 0$ when $x = 1, x = 3$



$\therefore$ The function is increasing in $]-\infty, 1[$ and $]3, \infty[$ but decreasing in $]1, 3[$

$\because f'(x) = 0$ when $x = 1 \in [0, 2]$, $x = 3 \notin [0, 2]$

$\because f(0) = -1$ absolute minimum & $f(1) = 3$ absolute maximum & $f(2) = 1$

3–a] If $f(x) = 4 + Cot\,x - Sec^2 x$, find the equation of the normal to the curve of the function f at a point lying on the curve and its x –coordinate equals $\frac{\pi}{4}$.

The Solution

$\because f(x) = 4 + Cot\,x - Sec^2 x$ when $x = \frac{\pi}{4}$ $\therefore f\left(\frac{\pi}{4}\right) = 4 + 1 - 2 = 3$

$f'(x) = -Cosec^2 x - 2\,Sec\,x\,.Sec\,x\tan x = -Cosec^2 x - 2\,Sec^2 x \tan x$

The slope of the tangent at $x = \frac{\pi}{4}$ $\therefore f'\left(\frac{\pi}{4}\right) = -2 - 2 \times 2 \times 1 = -6$

$\therefore$ The equation of the normal $\frac{y-y_1}{x-x_1} = \frac{-1}{m}$ $\quad \therefore \frac{y-3}{x-\frac{\pi}{4}} = \frac{1}{6}$

$\therefore 6(y-3) = 1\left(x - \frac{\pi}{4}\right)$ $\quad \therefore 6y - 18 = x - \frac{\pi}{4}$ $\quad \therefore x - 6y - \frac{\pi}{4} + 18 = 0$

3-b] An empty tank whose capacity is 10 cubic meters , if the water is poured gradually in that tank at a rate of (2 t + 3) cubic cm/m where t time in minutes , **find the time needed to fill the tank.**

The Solution

Let the volume of the tank is v $\quad \therefore \frac{dv}{dt} = 2t + 3$ $\quad \therefore v = \int (2t+3)\,dt = t^2 + 3t + c$

when the tank was empty $\therefore v = 0$ at t = 0 $\therefore c = 0$ $\therefore v = t^2 + 3t$

when the tank was fill $\therefore v = 10\,m^2$ $\quad \therefore 10 = t^2 + 3t$ $\quad \therefore t^2 + 3t - 10 = 0$

$\therefore (t-2)(t+5) = 0$ $\quad \therefore$ t = −5 , $\therefore$ t = 2 min. $\therefore$ The time needed to fill the tank = 2 min.

4-a] Find: $\lim_{x\to\infty} \left(\frac{2x-1}{2x+1}\right)^{2x}$

The Solution

$\lim_{x\to\infty} \left(\frac{2x-1}{2x+1}\right)^{2x} = \lim_{x\to\infty} \left(\frac{2x+1-2}{2x+1}\right)^{2x} = \lim_{x\to\infty} \left(1 + \frac{-2}{2x+1}\right)^{2x}$

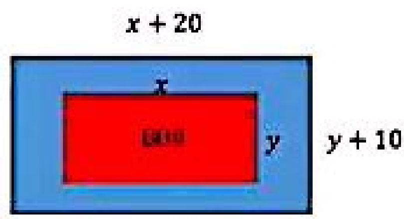
Let $y = \frac{-2}{2x+1}$ where $x \neq 0$, $y \to 0$ when $x \to \infty$

$\therefore 2xy + y = -2$ $\quad \therefore 2xy = -y - 2$ $\quad \therefore x = \frac{-y-2}{2y} = \frac{-y}{2y} + \frac{-2}{2y} = \frac{-1}{2} + \frac{-1}{y}$

$\lim_{x\to\infty} \left(1 + \frac{-2}{2x+1}\right)^{2x} = \lim_{y\to 0} (1+y)^{-1+\frac{-2}{y}} = \lim_{y\to 0} (1+y)^{-1} \times [\lim_{y\to 0} (1+y)^{\frac{-1}{y}}]^2 = 1 \times e^{-2} = e^{-2}$

4-b] A rectangle – like poster contains 800 cm^2 of the printed material where the widths of both lower and upper margins are 10 cm and the two side margins are 5cm. what are the two dimensions of the posters which make its area as minimum as possible .

The Solution



Let the dimensions of the printed material x cm. , y cm.

The dimension of the poster are $x + 20, y + 10$

$\therefore xy = 800$ $\quad \therefore y = \frac{800}{x}$

$\therefore$ The area of the poster = A = f(x) = (x + 20)(y + 10) = (x + 20) $\left(\frac{800}{x} + 10\right)$

$= 10x + \frac{16000}{x} + 1000$ $\therefore f'(x) = 10 - \frac{16000}{x^2}$ $\therefore f''(x) = \frac{32000}{x^3}$

Let $f'(x) = 0$ when $10 = \frac{16000}{x^2} \therefore x^2 = 1600$ $\therefore x = 40$

which make the area of the poster is minimum $\therefore y = 20$

$\therefore$ The dimension of the poster $= 40 + 20 = 60$ cm. and $20 + 10 = 30$ cm.

5-a] Find the volume of the solid generated by revolving the region bounded by the curve $y = 4 - x^2$ and the two positive parts of the axes of coordinates a complete revolution about x −axis.

The Solution

$\because y = 4 - x^2, y = 0$ to find the point of intersection $\therefore 4 - x^2 = 0$ $\therefore x^2 - 4 = 0$

$\therefore (x-2)(x+2) = 0 \therefore x = 2$ or $x = -2$ refused

$\therefore$ The area bounded by the two positive parts of x −axis

The rotation around x −axis $\therefore v = \pi \int_0^2 y^2\, dx = \pi \int_0^2 (16 - 8x^2 + x^4)\, dx$

$= \pi \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2 = \pi\left[32 - \frac{64}{3} + \frac{32}{5}\right] = \frac{256}{15}\pi$ cubic unit.

5-b] If $f(x) = x^3 + a\,x^2 + b\,x + 4$ where a and b are two constants,

Find the two values of a and b if the function f has a local minimum value when $x = 2$ and an inflection point when $x = 1$, then sketch the curve of the function f .

The Solution

$f(x) = x^3 + ax^2 + bx + 4 \therefore f'(x) = 3x^2 + 2ax + b \therefore f''(x) = 6x + 2a$

$\because$ at $x = 1$ there exists an inflection point $\therefore f''(1) = 0, \therefore 0 = 6 + 2a, \therefore a = -3$

$\because$ when $x = 2$ there exist local minimum $\therefore f'(2) = 0 \therefore 0 = 6 + 2x - 3 \times 2 + b, \therefore b = 0$

$\therefore f(x) = x^3 - 3x^2 + 4$, $\therefore f'(x) = 3x^2 - 6x = 3x(x-2) \therefore f'(x) = 0$ when $x = 0, x = 2$

$\therefore$ The function has critical point at $x = 0, x = 2$

The function is increasing in $]2, \infty[$ and $]-\infty, 0[$ But The function is decreasing in $]0, 2[$

$f(2) = 0$ local minimum value $\therefore$ The curve $f''(x) = 6x - 6 = 6(x-1)$ $\because f''(0) < 0$

$\therefore f(0) = 4$ *local maximum vaalue* $\because f''(2) > 0 \therefore$ of the function convex up in $]-\infty, 1[$ and the curve of the function convex down in $]1, \infty[$

$\therefore$ The point $(1, 2)$ is an inflection point the point of intersection with the two axis are $(3,0), (0,4), (-1,0)$ by adding the point $(3,4), (-2,-16)$

Test 6

1] In each of the following phrases, choose A if the phrase is true and B if the phrase is false.

1 – The local maximum value of the function is greater then its local minimum value (A) (B)

2 – The rate of change of $\sqrt{n^2+3}$ with respect to $\frac{n}{n+1}$ is : $\frac{n(n+1)^2}{\sqrt{n^2+3}}$ (A) (B)

3 – If $\sqrt{y} + \sqrt{x} = 2$, then $\frac{d^2y}{dx^2} = \frac{-1}{x\sqrt{x}}$ (A) (B)

4 – $\int \frac{x-4}{(x-2)^6}\,dx = \frac{(x-4)^2}{(x-2)^6} + c$ (A) (B)

5 – If $y = x\,Log_e x - x$, then $\frac{dy}{dx} = Log_e x$ (A) (B)

6 – If (a , f(a)) is an inflection point to the curve of the continuous function f , then:
f ''(a) = zero (A) (B)

The Solution

(1) (B) The absolute maximum for the function is greater than the absolute minimum

(2) (A) Let $y = \sqrt{t^2+3} = (t^2+3)^{\frac{1}{2}}$ $\therefore \frac{dy}{dt} = \frac{1}{2}(t^2+3)^{-\frac{1}{2}}(2t) = \frac{t}{\sqrt{t^2+3}}$

$z = \frac{t}{t+1}$, $\therefore \frac{dz}{dt} \frac{1(t+1)-t}{(t+1)^2} = \frac{1}{(t+1)^2}$ $\therefore \frac{dy}{dz} = \frac{dy}{dt} \div \frac{dz}{dt} = \frac{t}{\sqrt{t^2+3}} \div \frac{1}{(t+1)^2} = \frac{t(t+1)^2}{\sqrt{t^2+3}}$

(3)(B) $\because \sqrt{x} + \sqrt{y} = 2$ $\therefore \sqrt{y} = 2 - \sqrt{x}$ By squaring both sides $\therefore y = 4 - 4\sqrt{x} + x$

$\therefore y = 4 - 4x^{\frac{1}{2}} + x$ $\therefore \frac{dy}{dx} = -2x^{-\frac{1}{2}} + 1$ $\therefore \frac{d^2y}{dx^2} = x^{-\frac{3}{2}} = \frac{1}{x^{\frac{3}{2}}} = \frac{1}{x\sqrt{x}}$

(4)(B) $\int \frac{x-4}{(x-2)^6}dx = \int (x-4)(x-2)^{-6}\,dx = \int (x-2-2)(x-2)^{-6}\,dx$

$$= \int (x-2)^{-5}\,dx - \int 2(x-2)^{-6}\,dx = \frac{-1}{4}(x-2)^{-4} + \frac{-2}{-5}(x-2)^{-5} + c$$

$$= \frac{-1}{4}(x-2)^{-4} + \frac{2}{5}(x-2)^{-5} + c = \frac{1}{20}(x-2)^{-5}[-5(x-2) + 2\times 4] + c$$

$$= \frac{1}{20}(x-2)^{-5}[-5x+10+8] + c = \frac{1}{20}(x-2)^{-5}[-5x+2] + c$$

(5) (A) $\because y = xLog_e x - x$ $\therefore \frac{dy}{dx} = Log_e x + x \times \frac{1}{x} - 1 = Log_e x$

(6) (B) If (a , f(a)) is an inflection point to the curve of the continuous function f

$\therefore f''(x) = 0$ or $f''(a)$ not exists .

2–a] $\int \frac{7x^3}{2-5x^4}\,dx$ $\int 3e^{-5x} + \frac{\pi}{x}\ dx$

The Solution

$\int \frac{7x^3}{2-5x^4}dx = \frac{7}{-20}\int \frac{-20x^3}{2-5x^4}dx = -\frac{7}{20}Log_e|2-5x^4| + c$

$\int 3e^{-5x} + \frac{\pi}{x}\ dx = \frac{-3}{5}e^{-5x} + \pi\, Log_e|x| + c$

2–b] If $y = a\,e^{x^2+1}$ prove that : $\frac{d^3y}{dx^3} = 4xy(3+2x^2)$

The Solution

$\frac{dy}{dx} = (2xa)e^{x^2+1}$ $\therefore \frac{d^2y}{dx^2} = 2a\,e^{x^2+1} + (4a\,x^2)e^{x^2+1}$

L.H.S. = $\frac{d^3y}{dx^3} = 4axe^{x^2+1} + 8axe^{x^2+1} + 8ax^3e^{x^2-1} = 12axe^{x^2+1} + 8ax^3e^{x^2+1}$

$= 4x(ae^{x^2+1})[3+2x^2] = 4xy[3+2x^2] = R.H.S.$

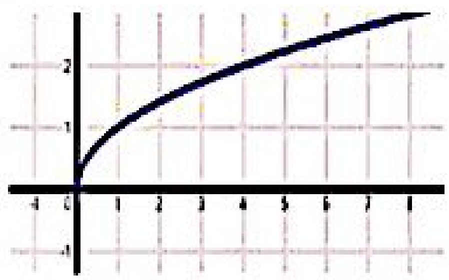
3–a] Find : $\int Cot\,x\ Csc^3x\ dx$

The Solution

$\int Cot\,x\ Cosec^3x\,dx = -\int (Cosec^2x)(-Cot\,x\ Cosec\,x)\,dx = \frac{-1}{3}Cosec^3x + c$

3–b] If S is the distance between point $(1,0)$ and point (x,y) lying on the curve $y = \sqrt{x}$ find the coordinates of point (x,y) at which S is as minimum as possible.

The Solution

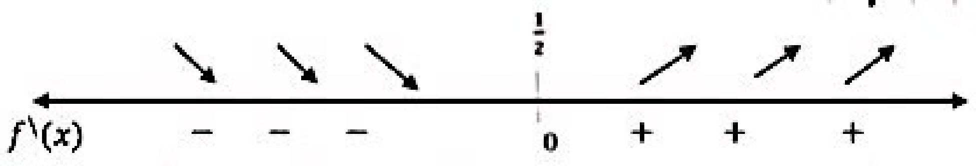


$S^2 = (x-1)^2 + (y-0)^2 = x^2 - 2x + 1 + y^2$

$\because y = \sqrt{x}$, $\therefore y^2 = x$

$\therefore S^2 = x^2 - 2x + 1 + x = x^2 - x + 1$

$\therefore S = (x^2 - x + 1)^{\frac{1}{2}}$ $\therefore \frac{ds}{dx} = \frac{1}{2}(x^2 - x + 1)^{-\frac{1}{2}}(2x-1)$



Let $\frac{ds}{dx} = 0$ $\therefore x = \frac{1}{2}$ at $x = \frac{1}{2}$ $\therefore S$ is local minimum and $y = \frac{1}{\sqrt{2}}$

$\therefore$ The coordinate of the point which is as minimum as possible is $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

4-a] Identify the absolute extrema values of the function f where $f(x) = |x|(x-4)$ in the intervals $[-1, 3]$

The Solution

$$f(x) = \begin{cases} x^2 - 4x & when\ 0 \le x \le 3 \\ -x^2 + 4x & when\ -1 \le x < 0 \end{cases} , \therefore f(0^+) = f(0) = f(0^-)$$

$\therefore$ The given function is continuous at $x = 0$

$\therefore$ The given function is continuous at $[-1, 3]$

$$f`(0^+) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{(x+h)^2-4(x+h)-[x^2-4x]}{h} \quad at \quad x = 0$$

$$\therefore f`(0^+) = \lim_{h\to 0^+} \frac{h^2-4h}{h} = \lim_{h\to 0^+} \frac{h(h-4)}{h} = \lim_{h\to 0^+} (h-4) = -4$$

$$f`(0^-) = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{-(x+h)^2+4(x+h)-[-x^2+4x]}{h} \quad at \quad x = 0$$

$$\therefore f`(0^-) = \lim_{h\to 0^-} \frac{-h^2+4h}{h} = \lim_{h\to 0^-} \frac{h(-h+4)}{h} = \lim_{h\to 0^-} (-h+4) = 4 , \because f`(0^+) \ne f`(0^-)$$

$\therefore$ The function is not differentiable at $x = 0$

$$\therefore f`(x) = \begin{cases} 2x - 4 & 0 \le x \le 3 \\ not\ exist & x = 0 \\ -2x + 4 & -1 \le x < 0 \end{cases}$$

$\therefore 2x - 4 = 0$ when $x > 0$ $\therefore x = 2 \in [-1, 3]$ and $-2x + 4 = 0$ when $x < 0$ $\therefore x = 2$

$f(-1) = -5$ absolute minimum $\qquad f(2) = -4 \qquad f(3) = -3$

$f(0) = 0$ absolute maximum

4-b] If the slope of the tangent to the curve $y = f(x)$ at any point on it equals $6x^2 + bx$ and $f(0) = 5$, $f(2) = -3$, find the value of the constant b, then sketch the curve of the function f.

The Solution

$\therefore y = \int (6x^2 + bx)\ dx = 2x^3 + \frac{1}{2}bx^2 + C$

$\because f(0) = 5 \quad \therefore C = 5 \quad \because f(2) = -3 \quad \therefore -3 = 16 + 2b + 5$

$\therefore b = -12 \qquad \therefore y = 2x^3 - 6x^2 + 5$

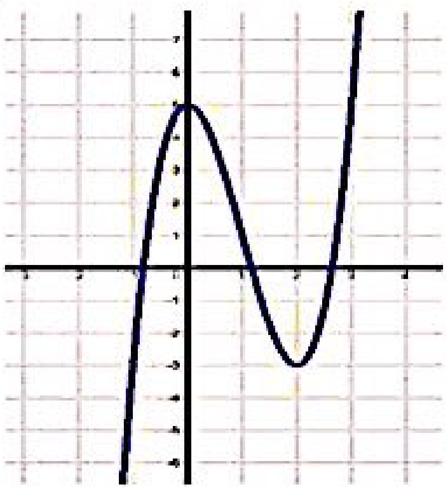
$\therefore f`(x) = 6x^2 - 12x = 6x(x-2)$

$\therefore f`(x) = 0$ when $x = 0, x = 2$ which are the critical point

$f``(x) = 12x - 12 , f``(0) = -12 < 0$

$\therefore f(0) = 5$ local maximum

$f``(2) = 12 > 0 . \therefore f(2) = -3$ local minimum



$\because f''(x) = 0$ when $x = 1$, $\because f''(1)$ change its sign before and after, $x = 1$

$\therefore (1, 1)$ is an inflection point

5–a] Find the rate of change of $Log_e(9 + x^3)$ with respect to $x^2 + 3$ and $x = 1$

The Solution

Let $y = log_e(g + x^3)$ $\quad \therefore \frac{dy}{dx} = \frac{3x^2}{9 + x^3}$ $\quad$ Let $z = x^2 + 3$ $\quad \therefore \frac{dz}{dx} = 2x$

$\therefore \frac{dy}{dz} = \frac{dy}{dx} \div \frac{dz}{dx} = \frac{3x^2}{9 + x^3} \div 2x$ at $x = 1$ $\quad \therefore \frac{dy}{dz} = \frac{3}{9+1} \div 2 = \frac{3}{20}$

5–b] If $a(0, 3), b(1, 4), c(2, 0)$, use the integration to find :

First : Surface area of ΔABC.

Second : the volume of the solid generated by revolving ΔABC a complete revolution about y–axis .

The Solution

The equation of $\overleftrightarrow{AB}$. $\quad \frac{y-3}{x-0} = \frac{4-3}{1-0}$

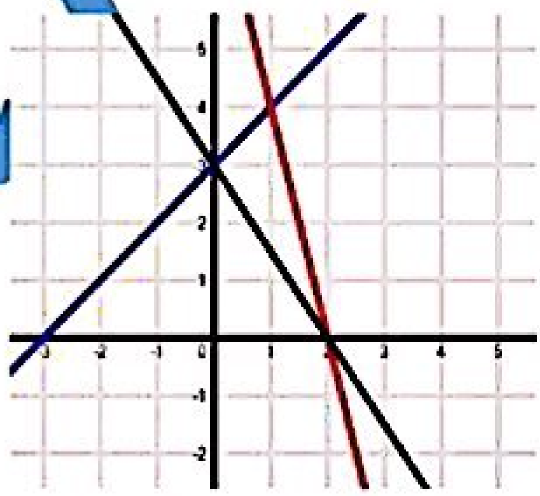
$\therefore \frac{y-3}{x} = 1$ $\quad \therefore y_1 = x + 3$

The equation of $\overleftrightarrow{BC} = \frac{y-0}{x-2} = \frac{4-0}{1-2}$

$\therefore \frac{y}{x-2} = -4$ $\quad \therefore y_2 = -4x + 8$

The equation of $\overleftrightarrow{AC}$ $\quad \frac{y-3}{x-0} = \frac{0-3}{2-0}$

$\therefore \frac{y-3}{x} = -\frac{3}{2}$ $\quad \therefore 2y - 6 = -3x$



$\therefore 2y = -3x + 6$, $\therefore y_3 = \frac{-3}{2}x + 3$

$\therefore$ Area of $\Delta = \int_0^1 (y_1 - y_3)\,dx + \int_1^2 (y_2 - y_3)\,dx = \int_0^1 \left(x + 3 - 3 + \frac{3}{2}x\right) dx +$
$\int_1^2 \left(8 - 4x - 3 + \frac{3}{2}x\right) dx = \int_0^1 \frac{5}{2}x\,dx + \int_1^2 \left(5 - \frac{5}{2}x\right) dx$
$= \left[\frac{5}{4}x^2\right]_0^1 + \left[5x - \frac{5}{4}x^2\right]_1^2 = \left(\frac{5}{4} - 0\right) + \left[(10 - 5) - \left(5 - \frac{5}{4}\right)\right] = \frac{5}{2}$ square unit

Test 7

1] In each of the following phrases, choose A if the phrase is true and B if the phrase is false.

1 – If $y^2 = 3x^2 - 7$, then : $\frac{dy}{dx} = \frac{y}{3x}$ (a) (b)

2 – The function $f : f(x) = x^3 - 3x + 1$ has an inflection point which is : $(0, 1)$ (a) (b)

3 – $\frac{d}{dx}[Cot(Cos\,3x)] = 3\,Sin3x\,Csc^2(Cos\,3x)$ (a) (b)

4 – $\int (1 - Cos\,x)^4 \,Sin\,x\; dx = \frac{1}{5}(1 - Cos\,x)^5 + C$ (a) (b)

5 – $\lim\limits_{x \to \infty}\left(1 + \frac{5}{x}\right)^x = e^5$ (a) (b)

6 – $\int \left(\frac{2e}{x} + \frac{x}{e}\right) dx = 2e\,Log_e|x| - \frac{x^2}{e} + C$ (a) (b)

The Solution

(1) (b) $y^2 = 3x^2 - 7$, $\therefore 2y\frac{dy}{dx} = 6x$, $\therefore \frac{dy}{dx} = \frac{6x}{2y} = \frac{3x}{y}$

(2)(a) $f(x) = x^3 - 3x + 1$ $\therefore f'(x) = 3x^2 - 3$ $\therefore f''(x) = 6x$ $\therefore f''(x) = 0$ when $x = 0$

$f''(x)$ change its sign before and after $x = 0$ at $x = 0$ $\therefore y = 1$ $\therefore (0, 1)$ is an inflection point

(3)(a) $\frac{d}{dx}[Cot(Cos\,3x)] = -Cosec^2(Cos\,3x) \times -3\,Sin\,3x = 3\,Sin\,3x\;Cosec^2(Cos\,3x)$

(4)(b)

(5)(a)

(6) (b) $2e\int \frac{dx}{x} + \frac{1}{e}\int x\,dx = 2e\,Log_e|x| + \frac{x^2}{2e} + c$

2–a] $\int x\,Sin\,x\;dx$ $\int_{-1}^{1} \sqrt{x^4 + x^2}\;dx$

The Solution

Let $u = x$, $Sin\,x\,dx = dv$, $du = dx$, $-Cos\,x = v$

$I = uv - \int v\,dx = -x\,Cos\,x + \int Cos\,x\;dx = -x\,Cos\,x + Sin\,x + C$

- $\sqrt{x^4 + x^2} = |x|\sqrt{x^2 + 1}$ the opposite figure represent the function

$\therefore \int_{-1}^{1} \sqrt{x^4 + x^2}\;dx = 2\int_0^1 x\sqrt{x^2 + 1}\;dx = \left[\frac{2}{3}(x^2 + 1)^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}\left[(2)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right] = \frac{2}{3}(2\sqrt{2} - 1)$

2-b] Find the equation of the tangent to the curve $y = Log_e\left(2 - \sqrt{2}\, Cos\, x\right)$ at the point lying On it and its x –coordinate equals $\frac{\pi}{4}$

.

The Solution

at $= \frac{\pi}{4}$, $\therefore y = log_e\left(2 - \sqrt{2} \times \frac{1}{\sqrt{2}}\right) = 0$, $\frac{dy}{dx} = \frac{\sqrt{2}\, Sin\, x}{2 - \sqrt{2}\, Cos\, x}$ at $x = \frac{\pi}{4}$ $\therefore \frac{dy}{dx} = \frac{\sqrt{2} \times \frac{\sqrt{2}}{2}}{2 - \sqrt{2} \times \frac{\sqrt{2}}{2}} = 1$

$\therefore$ **The equation of the tangent** $\frac{y - y_1}{x - x_1} = m$ $\therefore \frac{y - 0}{x - \frac{\pi}{4}} = 1$ $\therefore y = x - \frac{\pi}{4}$

3-a] Identify the convexity intravels downwards and the inflection points (if existed) to the curve of the function f where $f(x) = (x - 1)^4 + 3$

The Solution

$f'(x) = 4(x - 1)^3$ $\therefore f''(x) = 12(x - 1)^2$, Let $f''(x) = 0$ $\therefore x = 1$

[the curve is convex down] $\therefore$ **There is no inflection point**

$\because f''(x)$ does not change its sign before and after $x = 1$

3-b] A cuboid of metal whose base is square like . If the side length of the base increases at a rate of 0.4 cm/sec and the height decreases at a rate of 0.5 cm /c , find the rate of change of the volume when the side length of the base is 6cm and the height is 5cm.

The Solution

Let the dimensions of the square base are x, x & the dimension of the height is y

$x = 6$, $y = 5$, $\frac{dx}{dt} = 0.4$, $\frac{dy}{dt} = -0.5$

$\therefore$ The volume $= v = x^2 y$ **by derivative with respect to t**

$\therefore \frac{dv}{dt} = 2xy\frac{dx}{dt} + x^2\frac{dy}{dt} = 2 \times 6 \times 5 \times 0.4 + 36 \times -0.5 = 6\, cm^3$/sec.

4–a] If $f(x) = \int_0^3 x\sqrt{x+1} \quad dx$

The Solution

Let $y = x + 1 \therefore dy = dx$, $x = y - 1$ when $x = 0, \therefore y = 1$ and when $x = 3 \therefore y = 4$

$$\therefore I = \int_1^4 (y-1)y^{\frac{1}{2}}\,dy = \int_1^4 \left(y^{\frac{3}{2}} - y^{\frac{1}{2}}\right) dy = \left[\frac{2}{5}y^{\frac{5}{2}} - \frac{2}{3}y^{\frac{3}{2}}\right]_1^4 = \left(\frac{2}{3}\times 32 - \frac{2}{3}\times 8\right) - \left(\frac{2}{5} - \frac{2}{3}\right) = \frac{116}{15}$$

4–b] A rectangle – like playground in which two opposite sides end in a semi circle outside the rectangle of a diameter length equal to the length of this side. If the perimeter of the playground is 400 meters , prove that the surface area of the playground is as maximum as possible when the ground is a circle – like , then find its radius length.

The Solution

Let the length of the rectangle $= y$ m

The width of the rectangle

= the length of the diameter of the circle $= 2x$ m

$\therefore$ The radius of the circle $= x$ m

$\therefore$ The perimeter of the playground = 400 m

$\therefore 2x\pi + 2y = 400 \qquad \therefore y = 200 - \pi x$

$\therefore$ The area of the playground $= f(x) =$ area of the circle + area of rectangle

$= \pi x^2 + xy = \pi x^2 + x\,(400 - 2x\pi) = \pi x^2 + 400x - 2x^2\pi = 400x - \pi x^2$

$\therefore f\,'(x) = 400 - 2\pi x \qquad \therefore f\,''(x) = -2\pi \qquad$ Let $f\,'(x) = 0 \quad \therefore 400 - 2x\pi = 0 \quad \therefore x = \frac{200}{\pi}$

$\because f\,''\left(\frac{200}{\pi}\right) < 0 \quad \therefore$ at $x = \frac{200}{\pi}$ the area of the playground is maximum

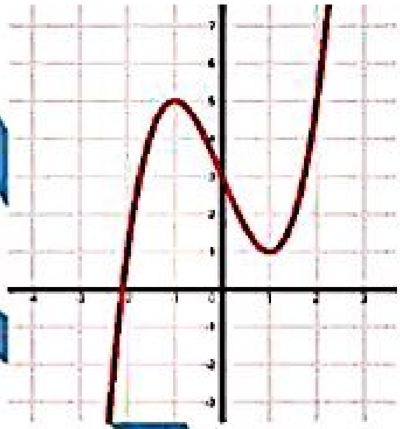
$\therefore y = 200 - \frac{200}{\pi} \times \pi = 0$ i.e. the playground in the shape of circle with radius $\frac{200}{\pi}\ m$

5-a] If $f(x) = x^3 - 3x + 3$, find :

First : the absolute extreme value of the function f in the interval f $[0, 2]$

Second: the area of the region bounded by the curve of the function f and the straight Lines $x = 0, x = 2, y = 0$

The Solution



$f(x) = x^3 - 3x + 3$

$\therefore f`(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$

Let $f`(x) = 0$

$\therefore x = 1 \in [0, 2]$, $x = -1 \notin [0, 2]$

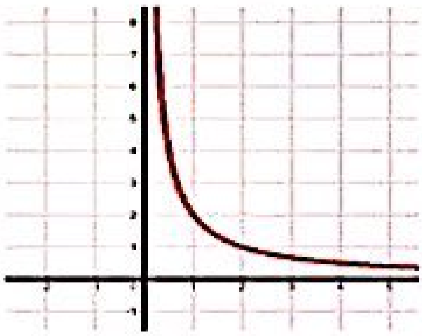
$\because$ f(0) = 3 , f(1) = 1 absolute min.

f(2) = 5 absolute max.

$A = \int_0^2 (x^2 - 3x + 3)\, dx = \left[\frac{1}{4}x^4 - \frac{3}{2}x^2 + 3x\right]_0^2 = 4 - 6 + 6 = 4$ square unit .

5-b] Find the volume of the solid generated by revolving the region bounded by the curve $xy = 2$ and the two straight lines $x = 1$ and $x = 2$

The Solution



$y = \frac{2}{x}$ $\because$ The rotation around x-axis

$\therefore v = \pi \int_1^2 y^2\, dx = \pi \int_1^2 \frac{4}{x^2}\, dx$

$= \pi \int_1^2 4x^{-2}\, dx = \pi[-4x^{-1}]_1^2 = \pi\left[\frac{-4}{x}\right]_1^2$

$= \pi(-2 + 4) = 2\pi$ cubic unit .

Test 8

1] Complete the following :

a] If $x^3y^2 = 1$, then : $[\frac{dy}{dx}]_{y=1} =$ ....

b] $\frac{d}{dx}[7e^{Sec\,x}] =$ ....

c] The function $f : f(x) = x^3 - 3x - 1$ has an inflection point which is : ....

d] If f is a continuous function on the interval $[2, 7]$,

then : $\int_2^7 f(x)\,dx + \int_7^4 f(x)\,dx =$ ....

e] The area of the region bounded by the two curves $y = x^2$ and $y = 4x$ equals .... squared units

f] If $y = x^2 \, Log_e \frac{x}{a}$, $a \neq 0$, then : $[\frac{d^3y}{dx^3}]_{x=4} =$ ....

The Solution

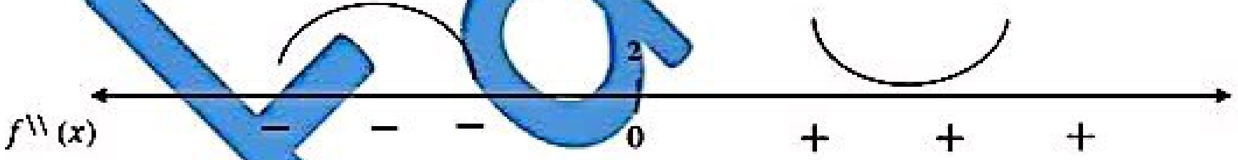
(a) $3x^2y^2 + x^3\left(2y\frac{dy}{dx}\right) = 0$ when $x = 1, y = 1$, $\therefore 3 + 2\frac{dy}{dx} = 0$ $\therefore 2\frac{dy}{dx} = -3$, $\frac{dy}{dx} = \frac{-3}{2}$

(b) $7\,Sec\,x\,tan\,x\,e^{sec\,x}$

(c) $f'(x) = 3x^2 - 3$ $\therefore f''(x) = 6x$ when $f''(x) = 0$ $\therefore x = 0$ $\therefore f(0) = -1$

$\because f''(x) < 0$ when $x < 0$, $f''(x) > 0$ when $x > 0$

$\because f''(x)$ *change its sign before and after* $x = 0$



$\therefore$ The point $(0, -1)$ is an inflection point

(d) $\int_2^7 f(x)\,dx - \int_4^7 f(x)\,dx = \int_2^4 f(x)\,dx + \int_4^7 f(x)\,dx - \int_4^7 f(x)\,dx = \int_2^4 f(x)\,dx$

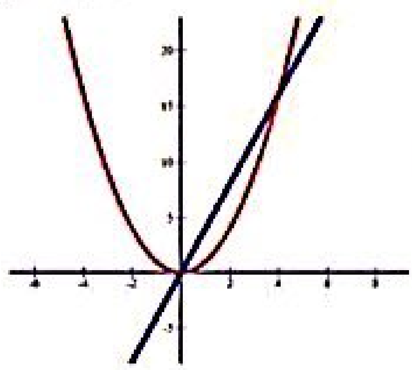
(e) $y_1 = x^2$, $y_2 = 4x$, Let $y_1 = y_2$ $\therefore x^2 = 4x$

$\therefore x^2 - 4x = 0$ $\therefore x(x-4) = 0$ $\therefore x = 0$ or $x = 4$

$\because y_2 \geq y_1$ for every $x \in [0, 4]$

$\therefore A = \int_0^4 (y_2 - y_1)\,dx = \int_0^4 (4x - x^2)\,dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^4$

$= \left[\left(32 - \frac{64}{3}\right) - 0\right] = \frac{32}{3}$ square unit .



(f) $\frac{dy}{dx} = 2x\,log_e\frac{x}{a} + x^2 \times \frac{a}{x} \times \frac{1}{a} = 2x\,log_e\frac{x}{a} + x$

$\frac{d^2y}{dx^2} = 2\,Log_e\frac{x}{a} + 2x \times \frac{a}{x} \times \frac{1}{a} = 2\,log_2\frac{x}{a} + 2$, $\frac{d^3y}{dx^3} = 2 \times \frac{a}{x} \times \frac{1}{a} = \frac{2}{x}$ $\quad \therefore \left[\frac{d^3y}{dx^3}\right]_{x=4} = \frac{1}{2}$

2-a] Find : $\int \frac{(x+3)^3 - 27}{x}\,dx$, $\int x^2 e^{-x}\,dx$

The Solution

$(x+3)^3 - 27 = (x+3-3)[(x+3)^2 + 3(x+3) + 9] = x[x^2 + 6x + 9 + 3x + 9 + 9]$

$= x(x^2 + 9x + 27)$

$\therefore I = \int (x^2 + 9x + 27)\,dx = \frac{1}{3}x^3 + \frac{9}{2}x^2 + 27x + c$

- Let $u = x^2$, $e^{-x}dx = dv$, $du = 2x$, $-e^{-x} = v$

$I = uv - \int v\,du = -x^2e^{-x} + \int 2xe^{-x}\,dx$

Let $u = 2x$, $e^{-x}\,dx = dv$, $dv = 2\,dx$, $-e^{-x} = v$

$\therefore I = -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + c = -e^{-x}[x^2 + 2x + 2] + c$

2-b] Find the equation of the tangent to the curve of the function f where $f(x) = 2\tan^3 x$ at the point lying on the curve of the function f and its $x-$ coordinate equals $\frac{\pi}{4}$

The Solution

$f(x) = 2tan^3 x$ $\quad \therefore f`(x) = 6\,tan^2 x\,Sec^2 x$ at $x = \frac{\pi}{4}$ $\quad \therefore f(x) = 2 \times (1)^3 = 2$

$f`\left(\frac{\pi}{4}\right) = 6 \times tan^2(45) \times Sec^2(45) = 12$,

$\therefore$ The equation of the tangent $\frac{y-y_1}{x-x_1} = m$ $\quad \therefore \frac{y-2}{x-\frac{\pi}{4}} = 12$

$\therefore y - 2 = 12\left(x - \frac{\pi}{4}\right)$ $\quad \therefore y - 2 = 12x - 3\pi$ $\quad \therefore y = 12x + 2 - 3\pi$

3-a] Find $\int_0^5 |x - 2|\,dx$

The Solution

$\int_0^2(-x+2)\,dx + \int_2^5(x-2)\,dx = \left[-\frac{1}{2}x^2 + 2x\right]_0^2 + \left[\frac{1}{2}x^2 - 2x\right]_2^5$

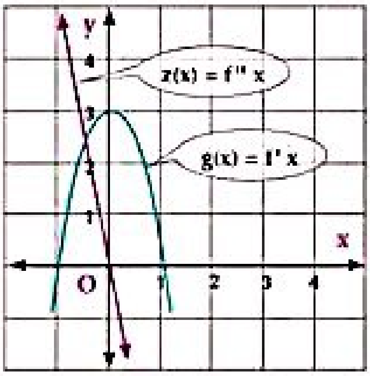
$= [(-2+4) - 0] + \left[\left(\frac{25}{2} - 10\right) - (-2+4)\right] = \frac{13}{2}$

3-b] The opposite figure shows the two curves of the two functions g and h where :

$g(x) = f'(x)$, $z(x) = f''(x)$ and

f is a polynomial function at the variable x.

Sketch the curve of f know that it passes through the two points $(-1, 0)$, $(1, 4)$



The Solution

$\because f(x)$ is a polynomial function

$f''(x)$ is a function of the 1st degree

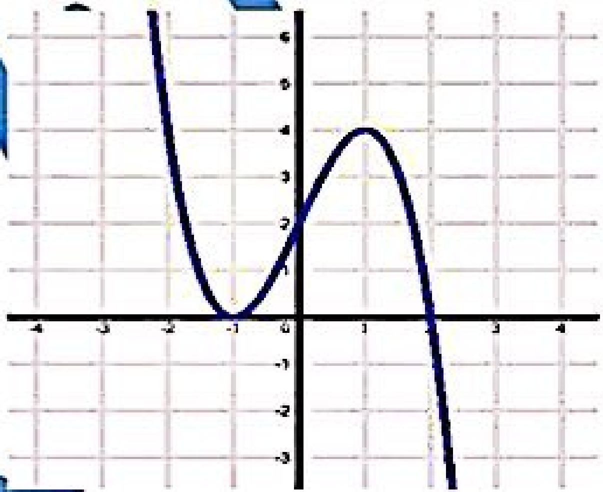
$f'(x)$ is a function of 2nd degree

$\therefore f(x)$ is a polynomial of 3rd degree

$\therefore f(x) = ax^3 + bx^2 + cx + d$

$\therefore f'(x) = 3ax^2 + 2bx + c$

$\therefore f''(x) = 6ax + 2b$



$\because$ The curve of $f''(x)$ passes through $(0, 0)$

$\therefore f''(0) = 0$ $\therefore 0 + 2b = 0$ $\therefore b = 0$ $\therefore f'(0) = 3$

$f'(1) = 0$ $\therefore 0 + 0 + c = 3$ $\therefore c = 3$ $0 = 3a + 2b + c$ $\because c = 3, b = 0$ $\therefore a = -1$

$\because$ The curve passes through $(-1, 0)$ $\therefore f(-1) = 0$ $\therefore 0 = -a + b - c + d$ $\therefore d = 2$

$\therefore f(x) = -x^3 + 3x + 2$ $\therefore f'(x) = -3x^2 + 3 = -3(x^2 - 1) = -3(x-1)(x+1)$ $\therefore f'(x) = 0$ when $x = 1, x = -1$ which the critical point for the function, the function is decreasing in $]-\infty, -1[$ and $]1, \infty[$ but the function is increasing in $]-1, 1[$ $\because f''(x) = -6x$

$\therefore f''(1) < 0$ $\therefore f(1) = 4$ local maximum value $\because f''(-1) > 0$

$\therefore f(-1) = 0$ local minimum value .

$\because f''(x) = 0$ when $x = 0$ $\because f''(x) > 0$ in $]-\infty, 0[$ $\therefore$ the curve is convex down

$\because f''(x) < 0$ in $]0, \infty[$

$\therefore$ The curve is convex up, $\because f''(x)$ change its sign before and after $x = 0$

$\therefore (0, 2)$ is an inflection point and the curve passes through $(1, 4)$

4-a] Identify the absolute extrema values of the function f in the interval [0 , 2] where $f(x) = 3\sqrt{4 - x^2}$

The Solution

$f(x) = 3(4 - x^2)^{\frac{1}{2}}$ $\quad \therefore$ $f`(x) = 3 \times \frac{1}{2}(4 - x^2)^{-\frac{1}{2}}(-2x) = \frac{-3x}{\sqrt{4-x^2}}$

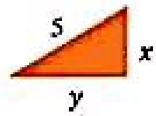
Let $f`(x) = 0 \therefore x = 0 \in [0, 2]$ $\therefore$ f`(x) is not exist when $4 - x^2 = 0$ when $x = 2 \in [0, 2]$

but $x = -2 \notin [-1, 1]$ refused , f(0) = 6 absolute maximum value

f(2) = 0 absolute minimum value .

4-b] A 5−meter rod is fixed by a hinge to the ground at its base . If its top rises up by a winch at a rate of 1m / m , find the rate of decreasing the projector length of the rod on the ground when the height of the top is 3 meters.

The Solution



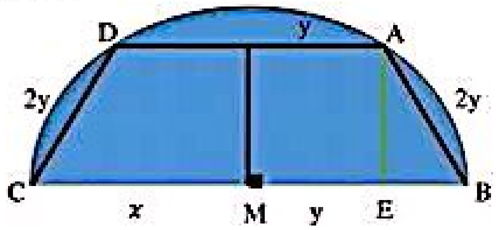
Let the height of one of the end of the rod about the floor = x m

and the length of its projection = y m from the graph $x^2 + y^2 = 25$

when $x = 3$ $\therefore y = 4$ $\therefore 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \therefore 2 \times 3 \times 1 + 2 \times 4 \times \frac{dy}{dt} = 0 \therefore \frac{dy}{dt} = -\frac{3}{4}$ m/min.

5-a] If a trapezoid is drawn in a semi−circle such that is base is the diameter of the semi−circle , determine the measure of the angle of the trapezoid base such that its area is as maximum as possible.

The Solution



Let the length of the small base of the trapezium = length of its sides = 2y unit length

The length of the greater base of the trapezium = the length of the diameter = 2 x unit length

The measure of the base of the trapezium is θ .

From the graph , BE = $x - y$

$\therefore H^2 = 4y^2 - (x - y)^2 = 4y^2 - [x^2 - 2xy + y^2] = 4y^2 - x^2 + 2xy - y^2 = 3y^2 - x^2 + 2xy$

$\therefore H = (3y^2 - x^2 + 2xy)^{\frac{1}{2}}$, the area of the trapezium = f(x) = $\frac{1}{2} \times (2x + 2y) \times H$

$\therefore f(x) = (x + y)[3y^2 - x^2 + 2xy]^{\frac{1}{2}}$

$f`(x) = 1[3y^2 - x^2 + 2xy]^{-\frac{1}{2}} + \frac{1}{2}[3y^2 - x^2 + 2xy]^{-\frac{1}{2}}[3y^2 - x^2 + 2xy + y^2 - x^2]$

$\therefore f`(x) = (3y^2 - x^2 + 2xy)^{-\frac{1}{2}}[4y^2 + 2xy - 2x^2] = \frac{4y^2+2xy-2x^2}{\sqrt{3y^2-x^2+2xy}}$

$f`(x) = 0$ when $4y^2 + 2xy - 2x^2$ $\therefore 2(2y - x)(y + x) = 0$, $\therefore x = 2y$ or $x = -y$ refused

BE = 0 , $\because f`(x) < 0$ when $2y < x$ $\therefore f`(x) > 0$ when $2y > x$

$\therefore x = 2y$ which make the area of the trapezium as maximum .

5–b] If a is the region bounded by the curve $xy = 4 + x^2$ and the straight lines $x = 1$, $x = 4$ and $y = 0$, find :

First: Area of region a in square units to the nearest unit .

Second : the volume of the solid generated by revolving the region a about x –axis .

The Solution

$y = \frac{4+x^2}{x} = \frac{4}{x} + x$

$A = \int_1^4 \left(\frac{4}{x} + x\right) dx = \left[4log_2\, x + \frac{1}{2}x^2\right]_1^4$

$= \left[4log_e\, 4 + \frac{1}{2} \times 4^2\right] - \left[2\, log_2 1 + \frac{1}{2}(1)^2\right]$

$= 4Log_2 4 + \frac{15}{2} \approx 13$ unit area

$\therefore v = \pi \int_1^4 y^2\, dx = \pi \int_1^4 \left(\frac{16}{x^2} + 8 + x^2\right) dx$

$= \pi \left[\frac{-16}{x} + 8x + \frac{1}{3}x^3\right]_1^4 = \pi\left[\left(-4 + 32 + \frac{64}{3}\right) - \left(-16 + 8 + \frac{1}{3}\right)\right] = 57\,\pi$ cubic unit .

Test 9

1] Choose the correct answer

1– If $x = 4n^2 + 7$, $y = \sqrt{n^3}$, $n = 1$, then : $\frac{dy}{dx}$ equals :

a] $\frac{3}{8}$ b] $\frac{3}{4}$ c] 2 d]6

2– The curve of the function f is convex downwards on if $f(x)$ equals :

a] $2 - x^2$ b] $2 + x^3$ c] $2 - x^4$ d] $2 + x^4$

3– If the cure of the function f : $f(x) = x^3 + k\,x^2 + 4$, $k \in R$ has an inflection point when $x = 2$, then k equals:

a] − 6 b] − 3 c] 6 d] 9

4– If f is a continuous function on RR, $\int_{-1}^{3} f(x)\ dx = 7$, $\int_{5}^{3} f(x)\ dx = -11$

, then : $\int_{-1}^{5} f(x)\ dx$ equals : ....

a] – 4 b] 18 c] 18 d] 77

5– $\int_{-1}^{3} |x - 1|\ dx$ equals :

a] – 6 b] 0 c] 4 d] 8

6– The area of the region bounded by the curve $y = x^3$ and the two straight lines $y = 0$ and $x = 2$ equals :

a] 1 b] 2 c] 4 d] 8

The Solution

(1) $\frac{dx}{dn} = 4n , \frac{dy}{dn} = \frac{3}{2}\sqrt{n}$ when $x = 1$ $\therefore \frac{dx}{dn} = 4 , \frac{dy}{dn} = \frac{3}{2}$ $\therefore \frac{dy}{dx}$ (at n = 1) = $\left(\frac{dy}{dn} \div \frac{dx}{dx}\right) = \frac{3}{8}$

(2) (d) $f(x) = 2 + x^4$ $\therefore f`(x) = 4x^3$, $f``(x) = 12x^2$ , $f``(x) > 0$ for every $x \in R$

$\therefore$ The curve is convex down on R .

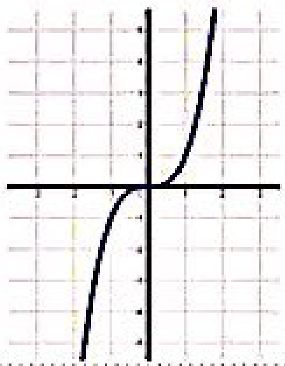
(3) $\because f(x) = x^3 + kx^2 + 4$, $\therefore f`(x) = 3x^2 + 2kx$, $f``(x) = 6x + 2k$

$\because$ The inflection point at $x = 2$ when $f``(2) = 0$ $\therefore 12 + 2k = 0$ $\therefore k = -6$

(4) $\int_3^5 f(x)\ dx = 11$, $\therefore \int_{-1}^5 f(x)\ dx = \int_{-1}^3 f(x)\ dx + \int_3^5 f(x)\ dx = 7 + 11 = 18$

(5) $$\int_{-1}^{3} |x-1|\, dx = \int_{-1}^{1} (-x+1)\, dx + \int_{1}^{3} (x-1)\, dx \left[-\frac{1}{2}x^2 + x\right]_{-1}^{1} + \left[\frac{1}{2}x^2 - x\right]_1^3$$

$$= \left[\left(-\frac{1}{2}+1\right) - \left(-\frac{1}{2}-1\right)\right] + \left[\left(\frac{9}{2}-3\right) - \left(\frac{1}{2}-1\right)\right] = 4$$



(6) $y = 0$ $\therefore x^3 = 0$, $\therefore x = 0$

$\therefore$ The limit of integration $x = 0$, $x = 2$

$\therefore A = \int_0^2 x^3\, dx = \left[\frac{1}{4}x^2\right]_0^2 = \frac{1}{4} \times 16 = 4$ square unit .

2-a] Find : $\int \frac{3x}{x^2-1}\, dx$, $\int 9x^2 e^{3x}\, dx$

The Solution

$I = \frac{3}{2}\int \frac{2x}{x^2-1}\, dx = \frac{3}{2} log_e |x^2 - 1| + C$

$u = 9x^2$, $e^{3x}\, dx = dv$, $du = 18x dx$, $v = \frac{1}{3}\, e^{3x}$

$I = u\,v - \int v du = 3\,x^2\, e^{3x} - \int\, 6x\, e^{3x}\, dx$

$u = 6x$, $e^{3x}\, dx = dv$, $du = 6dx$, $v = \frac{1}{3}\, e^{3x}$

$I = 3\,x^2\, e^{3x} - [\,u\,v\, - \int v du\,] =$

$= 3\,x^2\, e^{3x} - [\,2x\, e^{3x} - \int\, 2e^{3x} dx\,] = 3\,x^2\, e^{3x} - 2x\, e^{3x} + 2\int e^{3x}\, dx$

$= 3\,x^2\, e^{3x} - 2x\, e^{3x} + \frac{2}{3}\, e^{3x} + C$

2-b] Find the measure of the positive angle which the tangent of the curve $y^3 = x^2$ makes with the positive direction of x-axis when $x = 8$ to the nearest minute .

The Solution

$\because y^2 = x^2$ derivative each side by x $\therefore 3y^2 \frac{dy}{dx} = 2x$ $\therefore \frac{dy}{dx} = \frac{2x}{3y^2}$ at $x = 8$

$\therefore y^3 = 64$ $\therefore y = 4$ $\therefore \tan\theta = \left[\frac{dy}{dx}\right]_{(8,4)} = \frac{1}{3}$ $\therefore m(\angle\,\theta) = 18°$

3-a] If $\sin x = x\,y$, prove that : $x^2\,(\,y + y``) + 2\cos x = 2\,y$

The Solution

$\because \sin x = xy$ ----- (1) by derivative each side twice $\therefore \cos x = y + xy`$

$\therefore \cos x - y = xy`$ , $\therefore y` = \frac{1}{x}(\cos x - y)$ --------(2) , $- \sin x = y` + y` + xy``$, from (1)

$\therefore -xy = 2y` + xy``$ , from (2) $\therefore -xy = \frac{2}{x}(\cos x - y) + xy``$ multiply by x

$\therefore -x^2 y = 2\cos x - 2y + x^2 y``$ $\therefore x^2(y + y``) + 2\cos x = 2y$

3-b] If the curve $y = 2x^3 + 3x^2 + 4x + 5$ has two parallel tangent ; one of them touches the curve at point $(-1\ ,\ 2)$, **find the equation of the other tangent .**

The Solution

$\because y = 2x^3 + 3x^2 + 4x + 5 \quad \therefore \frac{dy}{dx} = 6x^2 + 6x + 4$ at $(-1\,,2) \quad \therefore \frac{dy}{dx} = 4$

$\because$ The two tangent are parallel $\therefore$ The slope of the other tangent $= 4$

$\therefore 6x^2 + 6x + 4 = 4 \quad \therefore 6x^2 + 6x = 0 \quad \therefore 6x\,(x+1) = 0 \quad \therefore x = -1 \quad x = 0$

By substitution in the equation of the curve at $x = -1$, $\therefore y = 2$

& at $x = 0 \therefore y = 2$ and this is the point of the other tangent the equation of the tangent

$\frac{y - y_1}{x - x_1} = m \quad \therefore \frac{y-5}{x-0} = 4 \quad \therefore y - 5 = 4x \quad \therefore 4x - y + 5 = 0$

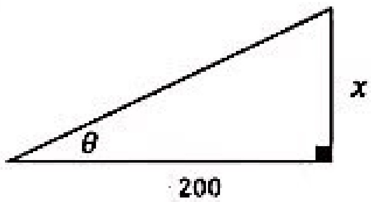
4-a] A balloon rises up vertically at a constant rate of 28 m/m. If the balloon is observed by a ground observer distant 200 m away from the site of launching the balloon , **find the rate of change of the angle of elevation of the observer when the balloon is 200m up.**

The Solution

Let x the height of the balloon from the floor after time t

θ is the angle of elevation of the top of the balloon

where x & θ are function of t from the opposite figure



$\tan\theta = \frac{x}{200}$ by derivative each side with respect to t

$\therefore Sec^2\theta \frac{d\theta}{dt} = \frac{1}{200} \times \frac{dx}{dt}$ when $x = 200$ m $\quad \therefore \theta = \frac{\pi}{4} \quad \therefore \theta = \frac{dx}{dt} = 28$ m/min.

$\therefore Sec^2\frac{\pi}{4} \times \frac{d\theta}{dt} = \frac{1}{200} \times 28 = \frac{28}{200} \quad \therefore 2 \times \frac{d\theta}{dt} = \frac{28}{200} \quad \therefore \frac{d\theta}{dt} = 0.07$ degree / min.

4-b] If the slope of the tangent to the curve of the function f at any point $(x\,,y)$ on the curve is $3\,(x^3 - 1)$, find **the local maximum** and **minimum** values to the curve of the function f and the inflection points if existed known that the curve passes thought point $(-2\,,-1)$, then sketch this curve.

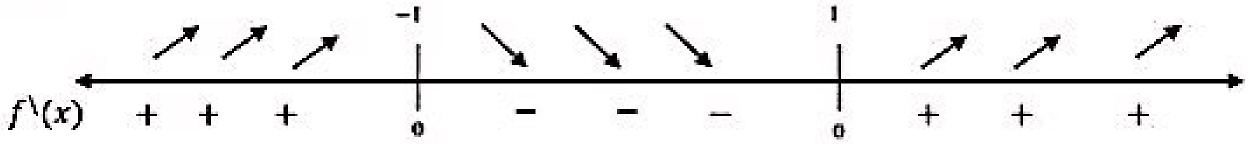
The Solution

$f`(x) = 3(x^2 - 1) = 3x^2 - 3 \qquad \therefore f(x) = \int (3x^2 - 3)\,dx = x^3 - 3x + C$

$\because$ The curve passes through $(-2\,,-1) \quad \therefore f(-2) = -1 \quad \therefore -1 = -8 + 6 + C \quad \therefore C = 1$

$\therefore f(x) = x^3 - 3x + 1 \qquad \therefore f\,`(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$

$\therefore f\,`(x) = 0$ when $x = 1$ *or* $x = -1$ which is the critical point at it

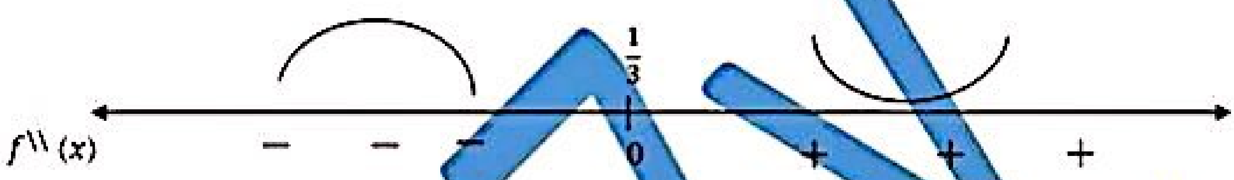


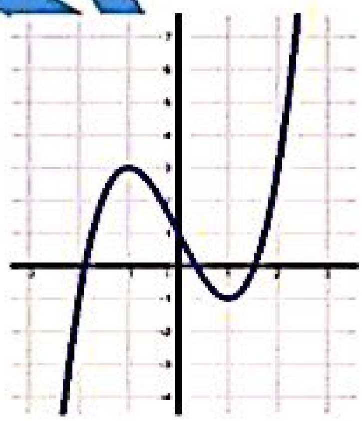
The function is increasing in $]-\infty, -1[$ and $]1, \infty[$

but the function is decreasing in $]-1, 1[$

$f\,``(x) = 6x \qquad \because f\,``(1) > 0 \qquad \therefore f(1) = -1$ is local minimum value

$\because f\,``(-1) > 0 \qquad \therefore f(-1) = 3$ local maximum value





$\because f\,``(x) = 0$ when $x = 0$ $\because f\,``(x) < 0$ in $]-\infty, 0[$

$\therefore$ The curve is convex up in this interval

$\because f\,``(x) > 0$ in $]0, \infty[$

$\therefore$ The curve is convex down in this interval

$\because f\,``(x)$ change its sign before and after $x = 0$

$\therefore (0, 1)$ is an inflection point and the curve passes through $(2, 3), (-2, -1)$.

••

5] The straight line $\overleftrightarrow{AB}$ intersects the curve of the function f at point C (x, y) where $x > 0$,

A $(0, 2)$, B $(6, 4)$ and $f(x) = \frac{9}{x}$, find :

a] The equation of the straight line $\overleftrightarrow{AB}$

b] The coordinates of point C

c] The equation of the normal on the curve of f at point C and prove that it passes through the origin point O .

d] The volume of the solid generated by revolving the region bounded by the normal $\overleftrightarrow{OC}$ and the curve of the function and the straight line $x = 6$ a complete revolution about x −axis .

The Solution

(a) The equation of $\overleftrightarrow{AB}$ is $\frac{y-2}{x-0} = \frac{4-2}{6-0}$ $\therefore \frac{y-2}{x} = \frac{1}{3}$ $\therefore x = 3y - 6$ ----- (1)

(b) $\because y = \frac{9}{x}$ by substitution in (1) $\therefore y = \frac{9}{3y-6}$ $\therefore 3y^2 - 6y + 9$ $\therefore 3[y^2 - 2y + 3] = 0$

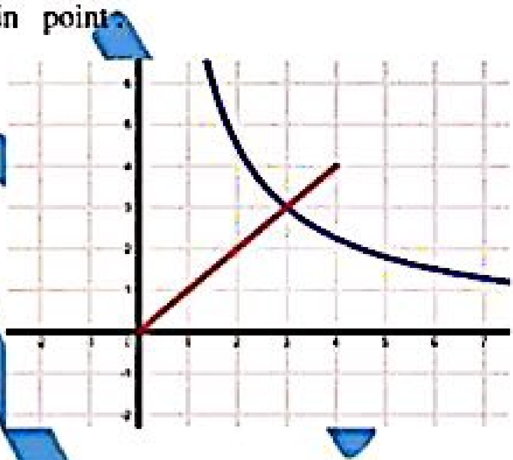
$\therefore 3(y-3)(y+1) = 0$ $\therefore y = -1$ $\therefore x = -9$ refused because $x > 0$

$\therefore y = 3$ $\therefore x = 3$ $\therefore$ The coordinate of the point $C = (3, 3)$

(c) $\frac{dy}{dx} = -\frac{9}{x^2}$ $\therefore$ The slope of the tangent at the point c $= \left(\frac{dy}{dx}\right)$ at $(3, 3) = -1$

$\therefore$ The slope of the normal at c = 1 $\therefore$ The equation of the normal at c is $\frac{y-3}{x-3} = 1$

$\therefore y = x$ $\therefore$ The normal is at c passes through the origin point.



(d) $v = \pi \int_0^3 y_1^2 \, dx + \pi \int_3^6 y_2^2 \, dx$

$$= \int_0^3 x^2 \, dx + \pi \int_3^6 \frac{81}{x^2} dx$$

$$= \pi \left[\frac{1}{3} x^3\right]_0^3 + \pi \left[-\frac{81}{x}\right]_3^6$$

$= \pi(9 - 0) + \pi\left(-\frac{81}{6} + \frac{81}{3}\right) = \frac{45}{2}\pi$ cubic unit.

Test 10

1] Complete the following :

a] $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x+3} =$ ....

b] $\frac{d}{dx}(5 - 2 \cot x)^3 =$ ....

c] If the function $f : f(x) = k\,x^3 + 9x^2$ has an inflection point when $x = -1$, then : k = ...

d] $\int_{-1}^{3} (4x^3 - 6x^2 + 5) \, dx =$ ....

e] if f is a continuous function on the interval [1 , 4] , then $\int_1^4 f(x) \, dx + \int_4^1 f(x) \, dx$ = ...

f] The area of the region bounded by the two curves $y = x^4 + 1$ and $y = 2x^2$ equals .... square units.

The Solution

a] $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x+3} = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x \times \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^3 = e \times (1 + 0)^3 = e$

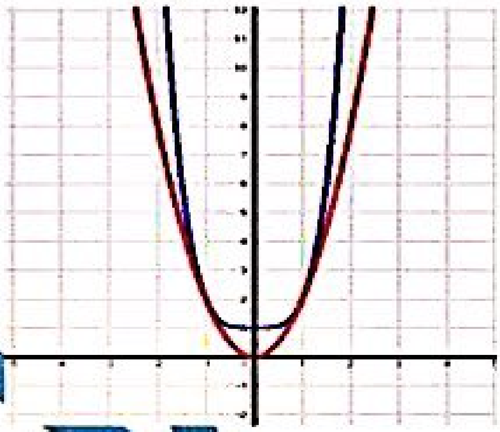
(b) $3[5 - \cot x]^2 \times (-2\,Cosec^2\,x) = -6\,Cosec^2\,x\,(5 - \cot x)^2$

(c) $f'(x) = 3kx^2 + 18x$, $f''(x) = 6kx + 18$ $\because$ The curve an inflection point at $x = -1$

$\therefore f''(-1) = 0$ $\therefore -6k + 18 = 0$ $\therefore k = 3$

(d) $[x^4 - 2x^3 + 5x]_{-1}^{3} = (81 - 54 + 15) - (1 + 2 - 5) = 34$

(e) $\because \int_4^1 f(x)\, dx = -\int_1^4 f(x)\, dx$ $\therefore$ The value = zero



(f) $y_1 = x^4 + 1$, $y_2 = 2x^2$, Let $y_1 = y_2$

$\therefore x^4 + 1 = 2x^2$, $\therefore x^4 - 2x^2 + 1 = 0$

$\therefore (x^2 - 1)^2 = \text{zero}$, $\therefore (x - 1)(x + 1) = 0$

$\therefore x = 1, x = -1$, $\because y_1 \geq y_2$ for every $x \in [-1, 1]$

$$\therefore A = \int_{-1}^{1} (y_1 - y_2)\, dx = 2\int_0^1 (y_1 - y_2) dx$$

$= 2\int_0^1 (x^4 + 1 - 2x^2)\, dx = 2\left[\frac{1}{5}x^5 + x - \frac{2}{3}x^3\right]_0^1 = 2\left[\left(\frac{1}{5} + 1 - \frac{2}{3}\right) - (0)\right] = \frac{16}{5}$ square unit

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2-a] Find : $\int \tan(3x + 1)\ dx$, $\int (1 - x^2)(3x - x^3)^5\ dx$

The Solution

$\int \frac{\text{Sin}(3x+1)}{\text{Cos}(3x+1)} dx = -\frac{1}{3}\int \frac{-3\,\text{Sin}(3x+1)}{\text{Cos}(3x+1)} dx = -\frac{1}{3} log_2|\text{Sin}(3x + 1)| + C$

$\frac{1}{3}\int (1 - x^2)(3x - x^3)^5\ dx = \frac{1}{3} \times \frac{1}{6}(3x - x^3)^6 + C = \frac{1}{18}(3x - x^3)^6 + C$

---

2 – b] If the two parametric equations of the function f where $y = f(x)$ are : $x = 2n^3 + 3$ and $y = n^4$, find each of the following when $n = 1$:

First : The equation of the tangent to the curve of the function f

Second : $\frac{d^2y}{dx^2}$

The Solution

$\because x = 2n^3 + 3$, $\therefore \frac{dx}{dn} = 6n^2$, $\because y = n^4$, $\therefore \frac{dy}{dn} = 4n^3$, $\therefore \frac{dy}{dx} = \frac{dy}{dn} \div \frac{dx}{dn} = \frac{2}{3}n^2$,

First : the slope of the tangent $= \frac{dy}{dx}\ (at\ \ n = 1) = \frac{2}{3}$, $\because x = 5, y = 1$ when $n = 1$

$\therefore$ The equation of the tangent $\frac{y - y_1}{x - x_1} = m$ $\therefore \frac{y-1}{x-5} = \frac{2}{3}$ $\therefore 2(x - 5) = 3(y - 1)$

$\therefore 2x - 10 = 3y - 3$ $\therefore 2y - 3y - 7 = 0$

Second : $= \frac{d}{dn}\left(\frac{dy}{dx}\right) \times \frac{dn}{dx} = \frac{4}{3}n \times \frac{1}{6n^2} = \frac{1}{9n}$ $\qquad \therefore \frac{d^2y}{dx^2}[(at\ n = 1)] = \frac{1}{9}$

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3 – a] Investigate the convexity of the curve of the function f where $f(x) = |x^3 - 1|$ and show the inflection points if existed.

The Solution

$f(x) = \begin{cases} x^3 - 1 & x > 1 \\ -x^3 + 1 & x < 1 \end{cases}$, $\because f(1^+) = f(1^-) = f(1) = 0$ $\therefore f(x)$ **is continuous at $x = 1$**

$\therefore$ **f(x) is continuous on R** , $f\,`(1^+) = \lim_{h\to0^+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to0^+} \frac{[(x+h)^3-1]-[x^3-1]}{h}$ at $x = 1$

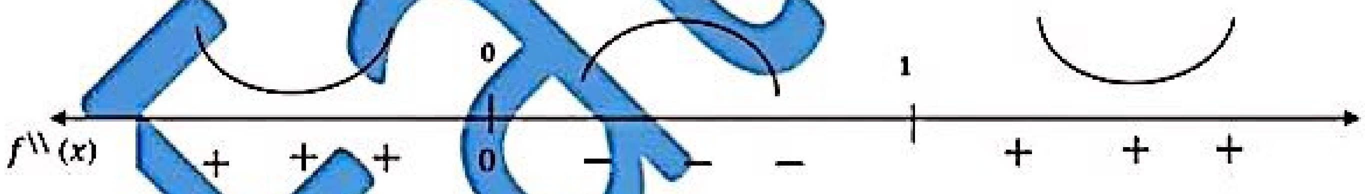
$\therefore f\,`(1^+) = \lim_{h\to0^+} \frac{(1+h)^3-1}{h} = \lim_{h\to0^+} \frac{(1+h)^3-(1)^3}{(1+h)-(1)} = \frac{3}{1}(1)^2 = 3$

$f\,`(1^-) = \lim_{h\to0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to0^-} \frac{[-(x+h)^3+1]-[-x^3+1]}{h}$ at $x = 1$

$\therefore f\,`(1^-) = \lim_{h\to0^-} \frac{-(1+h)^3+1}{h} = \lim_{h\to0^-} \frac{-(1+h)^3+1}{-(1+h)+1} \times \frac{-(1+h)+1}{h} = 3 \times -1 = -3$

$\therefore$ **$f\,`(1^+) = f\,`(1^-)$ $\therefore f(x)$ is not differentiable at $x = 1$** $\therefore f\,`(x) = \begin{cases} 3x^2 & x > 1 \\ not\ exist & at\ x = 1 \\ -3x^2 & x < 1 \end{cases}$

$f\,``(x) = \begin{cases} 6x & x > 1 \\ -6x & x < 1 \end{cases}$ $\quad \because f\,``(x) = 0$ when $x = 0$ at $x < 0$ $\therefore f\,``(x) > 0$



$\because f\,``(x) > 0$ when $-\infty < x < 0$ $\therefore$ **The curve is convex down in $]1, \infty[$**

$\therefore f\,``(x) < 0$ when $0 < x < 1$ $\therefore$ The curve is convex up in $]0, 1[$

$\because f\,``(x) > 0$ when $1 < x < \infty$ $\therefore$ **The curve is convex down in $]1, \infty[$**

$\because f\,``(1)$ is not exist at $x = 1$, $\therefore$ There is no tangent for the curve at the point (1 , 0)

$\therefore$ The point (1 , 0) is not an inflection point

---

3 – b] If $\int_{-2}^{3} f(x)\,dx = 9$, $\int_{5}^{3} f(x)\,dx = 4$, find the value of $\int_{-2}^{5}[3f(x) - 6x]\,dx$

The Solution

$\because \int_5^3 f(x)\,dx = 4$ $\qquad \therefore \int_3^5 f(x)\,dx = -4$ $\qquad \because \int_{-2}^3 f(x)\,dx = 9$

$\therefore \int_{-2}^5 f(x)\,dx = \int_{-2}^3 f(x)\,dx + \int_3^5 f(x)\,dx = 9 - 4 = 5$

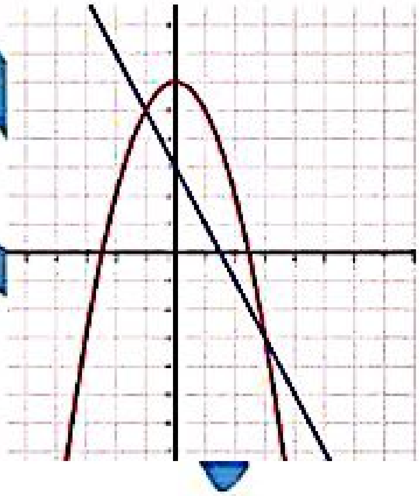
$\therefore \int_{-2}^{5}[3f(x)-6x] = 3\int_{-2}^{5} f(x)\,dx - 6\int_{-2}^{5} x\,dx$

$= 3\times 5 - 6\left[\frac{1}{2}x^2\right]_{-2}^{5} = 15 - 6[\frac{25}{2} - 2] = 15 - 75 + 12 = -48$

4 – a] Find the area of the plane region bounded by the two curves

$y + x^2 = 6$, $y + 2x - 3 = 0$

The Solution



$y_1 = 6 - x^2$, $y_2 = 3 - 2x$. Let $y_1 = y_2$

$\therefore 6 - x^2 = 3 - 2x \quad \therefore x^2 - 2x - 3 = 0$

$\therefore (x-3)(x+1) = 0 \quad \therefore x = 3 , x = -1$

$\because y_1 \geq y_2$ for every $x \in [-1 , 3]$

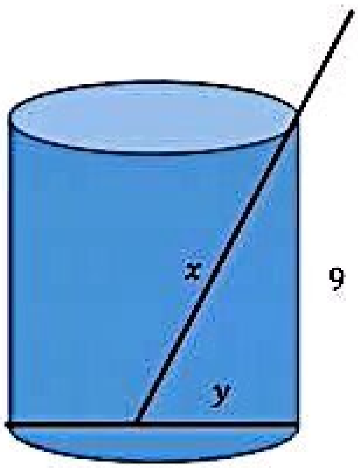
$\therefore A = \int_{-1}^{3}(y_1 - y_2)\,dx = \int_{-1}^{3} 6 - x^2 - 3 + 2x\,dx$

$= \int_{-1}^{3}(3 - x^2 + 2x)\,dx = \left[3x - \frac{1}{3}x^3 + x^2\right]_{-1}^{3}$

$= (9 - 9 + 9) - \left(-3 - \frac{1}{3} + 1\right) = \frac{34}{3}$ square unit .

4 – b] A right circular cylinder– like container of interior height 9cm and the interior radius length of its base is 6cm. A 16 cm metal rod is placed in the container. If the rate of sliding the rod away from the edge of the cylinder is 2cm/sec , find the rate of sliding the rod on the cylinder base when the rod reaches the end of its base.

The Solution



Let the part of the square unit sliding the rod away from the edge of the cylinder = x cm ,

the distance between the rod and the cylinder = y cm

From the geometry shape

$x^2 = 81 + y^2$ ------ (1)

when the rod reach to the end of the base of the cylinder

$\therefore$ y = 12 cm. by substitution $\therefore x = 15$ cm. derivative with respect to n

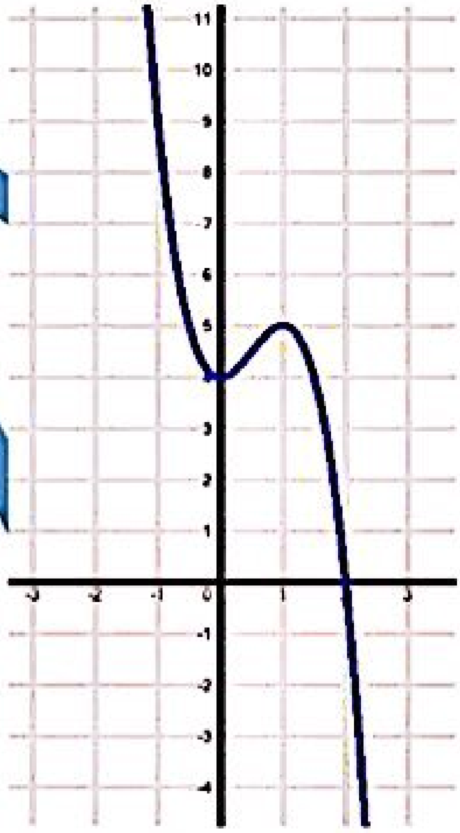
$\therefore 2x\frac{dx}{dn} = 2y\frac{dy}{dn} \quad \therefore 2\times 15\times 2 = 2\times 12\times \frac{dy}{dn} \quad \therefore \frac{dy}{dn} = \frac{5}{2}$ cm/sec.

5–a] If the rate of change of the slope of the tangent to a curve at any point (x, y) on it is $6(1-2x)$ on it is $x = 1$ and the curve has a critical point when $x = 1$ and the function has a local minimum value equals 4.

First: **Find the equation of the normal to the curve when $x = -1$**

Second : Sketch the curve of the function and show the maximum and minimum values and the inflection points if existed.

The Solution



$\because f``(x) = 6(1-2x) = 6 - 12x$

$\therefore f`(x = \int(6-12x)\ dx = 6x - 6x^2 + c$

$\because$ The curve has critical point at $x = -1$

$\therefore f`(-1) = 0, \therefore 0 = 6 - 6 + c, \therefore c = 0$

$\therefore f`(x) = 6x - 6x^2 = 6x(1-x), f`(x) = 0$

when $x = 0, x = 1$ $\because f``(0) > 0$

$\therefore$ at $x = 0$ there exists local minimum value = 4 (given)

$\therefore$ The curve passes through $(0, 4)$

$\therefore f(x) = \int(6x - 6x^2)\ dx = 3x^2 - 2x^3 + c$

$\because$ The curve passes through $(0, 4)$ $\therefore f(0) = 4$

$\therefore 4 = 0 - 0 + c, \therefore c = 4, \therefore f(x) = 3x^2 - 2x^3 + 4$

$f`(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$

$\therefore f`(x) = 0$ when $x = 0, x = 1$

The function is decreasing in $]1, \infty[$ and $]-\infty, -1[$

but the function is increasing in $]0, 1[$, $f``(x) = 6 - 12x$, $\because f``(0) > 0$

$\therefore f(1) = 4$ is local minimum value $, \because f``(1) < 0 \therefore f(1) = 5$ local maximum value

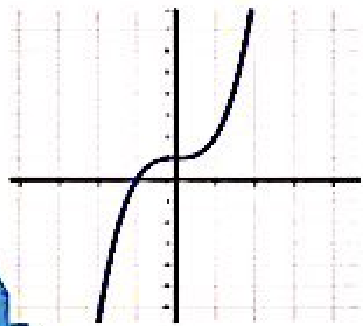
$\because f``(x) = 0$ when $= \frac{1}{2}, \because f``(x) > 0$

$\therefore in\ ]-\infty, \frac{1}{2}[$, the curve is convex down in this interval , $f``(x) < 0$ in $]\frac{1}{2}, \infty[$

$\therefore$ The curve is convex up in this interval the point $\left(\frac{1}{2}, \frac{9}{2}\right)$ is an inflection point

5 – b] Find the volume of the solid generated by revolving the plane region bounded by the curves : y = $x^3 + 1$, $y = 0$ and $x = 0$, $x = 1$ a complete revolution about x –axis.

The Solution



The limit of integration are $x = 0 , x = 1$

and the region above x –axis y = 0

the rotation around x –axis

$$\therefore v = \pi \int_0^1 y^2 \, dx = \pi \int_0^1 (x^6 + 2x^3 + 1) \, dx$$

$$= \pi \left[\frac{1}{7}x^7 + \frac{1}{2}x^4 + x\right]_0^1 = \pi\left[\left(\frac{1}{7} + \frac{1}{2} + 1\right) - (0)\right] = \frac{23}{14}\pi \text{ cubic unit .}$$

With my best wishes